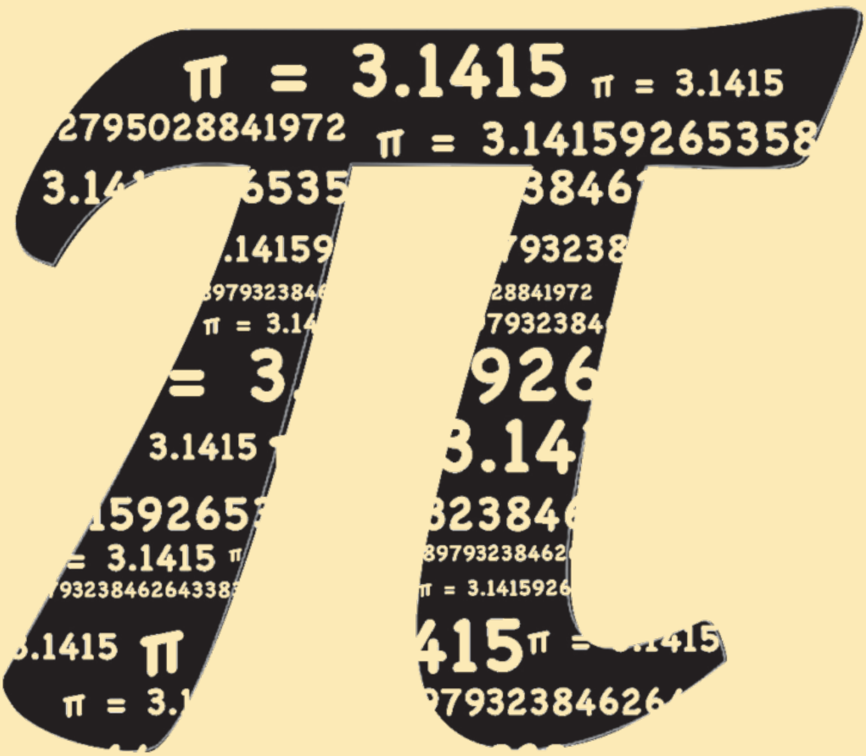
RECENT ADVANCES IN MATHEMATICS

EDITED BY
Assist. Prof. Esen HANAÇ DURUK



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PREFACE

Our book is titled Recent Advances in Mathematics. The book primarily discusses the new methodologies, including statistical analysis, theoretical analysis, modified method analysis, and comparisons between the new and traditional methodologies. Each chapter has certain techniques for summing up the issues that clarify the ideas. Each topic is presented with care and it is hoped that the book will satisfy the needs of scholars from various professional fields. For their particular interests, the research in this field is therefore very significant and helpful. I'd like to thank the authors for their invaluable works and also the publishing house for their cooperation.

Assist. Prof. Esen HANAÇ DURUK

CHAPTER 1

A STUDY ON THE NUMBER π AND RANDOMNESS

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INTRODUCTION

The number π is one of the most elegant flowers in the mathematical garden. It has been a flower that mathematicians and other scientists have smelled with curiosity and interest for hundreds of years since Archimedes (Öztürk & Özbek, 2015). This number has many features: It is a transcendent number, that is, it is a number that cannot be the root of a polynomial whose coefficients are integers. The proof of this was made by Ferdinand von Lindemann in 1882. His proof was based on two centuries of important mathematical contributions. The $e^{i\pi}+1=0$ equation, which we encounter in the mathematics literature and whose aesthetic properties are often mentioned by mathematicians, is quite interesting in that it contains $e, i, \pi, 1, ve\ 0$ numbers, which are the most important constant numbers of mathematics. Although many different methods are used to calculate π, today convergent infinite series, multiplications and sequential recurrence relations are used (Borwein, 2000).

For thousands of years, people have been trying to calculate more decimal places of π, and it is a matter of curiosity how these decimal places are distributed. Where does this interest in π come from? What other properties of π are ready to be discovered other than those known so far? The most elegant flower of the mathematical garden stands there and perhaps waits like a lover ready to offer its infinite features. In almost all mathematics books, especially those written for people who are interested in mathematics, the properties of π are mentioned. It's really interesting to see how π is used differently in geometry, probability, differential and integral calculations (Özbek, 2018).

Why would anyone want to calculate the value of π to billions of digits, as is done today with supercomputers? What is the source of this interest in the decimal places of π? This is used to measure the capabilities of supercomputers' hardware and software. Computational methods can lead to new ideas and concepts. Doesn't π have any order or pattern? Does it contain an endless variety of patterns? Are some numbers in π more common? Aren't these numbers randomly distributed? Perhaps the interest and admiration that mathematicians have felt for π throughout the centuries can be compared to the strong desires and emotions that drive mountain climbers to climb higher and higher.

In fact, the answers to all these questions have not been given clearly yet. A new research article about π is published every day. As long as human curiosity and passion continues, the desire to find a new aesthetic direction in π seems to continue forever. This study deals with the decimal digits of π within the framework of the concept of random sequence, and aims to discuss the suitability of these decimal digits for the definition of random sequence. Also given are examples of how to simulate using these decimal places.

1. RANDOMNESS, MODELLING AND SIMULATION

Various experiments are carried out to test the validity of the models, but in some cases it can be quite difficult and expensive to conduct experiments. In such cases, simulation processes are performed, which means experimenting on the model. Simulation is a word that means imitation. Nowadays it is used as a buzzword. The two most powerful tools of the human mind in modeling are mathematics and statistics. Statistics comes to the fore especially in modeling phenomena involving randomness. In this case, the question of what "randomness" is is important. The "randomness" debate will not be entered here. "Ragelessness" is one of the main topics discussed in the fields of science, philosophy and art. The articles in the bibliography can be used to obtain detailed information about this concept.

Random numbers are used in the simulation phase in many applied science fields such as Econometrics, Numerical Analysis, Cryptography, Computer Programming, Experimental Physics, Statistics, etc. Simulation; It is the experimentation of random events, systems and processes on a model in a computer environment (Morgan, 1992). In recent years, simulation is one of the leading methods used especially in the field of education. Random numbers are the basis of the simulation. It is desired that the simulation process should be able to imitate the event in the real world well; if the imitation cannot be done well, the experiment will not be able to represent the event in the real world well. For these reasons, the concept of random sequence is of great importance in terms of application. Various functions of random variables with uniform distribution in the cccc range are used to obtain observations from statistical distributions. If random numbers cannot be generated from a uniform distribution in the ccc range, naturally it will not be possible to generate numbers from other distributions. For this purpose, various generators (functional relationships) are used and generators that provide various statistical properties are used as random number generators. These numbers are known as "pseudo-random numbers" because they are generated according to certain mathematical rules. Many methods are used to generate these numbers in the computer environment. The most important of these methods are linear conjugate generators (Deak, 1990).

These,

x(k+1)=a*x(k)+c (mod m)

It is in the form. Numbers are produced sequentially by determining the starting value of x(0), the numbers a, c and m. The important thing here is that the numbers do not repeat each other and can be produced as many times as desired. These produced numbers are also required to meet some statistical properties. Numbers generated in this way are called “pseudo-random” numbers. In the computer, we see the initial value of x(0) as a random number because we do not know the numbers a, c, m and they pass some tests. True random numbers are generated as a result of experiments involving randomness or by electronic means. For convenience, these pseudo-random numbers are used in the computer. Generating these random numbers is a separate field of expertise.

2. CALCULATION OF π USING RANDOM NUMBERS

π can also be calculated using random numbers. Let's consider the square of -1 x 1 and -1 y 1 in the (x,y) coordinate system and call this region S. Let's consider the unit circle in this coordinate system again. Let's call the region formed by the unit circle A. Let's shoot random arrows into this square. The arrows fall inside the unit circle only when x^2+y^2 1 equality is achieved. Since the area of the square is 4 and the area of the circle is π, the probability of a randomly thrown arrow falling inside the unit circle is found as P(A)=Area(A)/Area(S)= π/4. We can calculate the P(A) value by shooting arrows or using random numbers. Arrows are often thrown into this square and those that fall into the circle are counted. If this is called the number of successes, the ratio of the total number of successes to the number of attempts gives us an idea about this probability. If the total number of successes is m and the number of attempts is n, π=4*m/n is found from the equation m/n= π/4. Since the number of trials is certain, just finding the number m will do the trick. From these results, the π value can be approximately calculated. In the Fig. 1 arrows falling inside the square are shown in green, and those falling inside the circle are shown in pink.

π

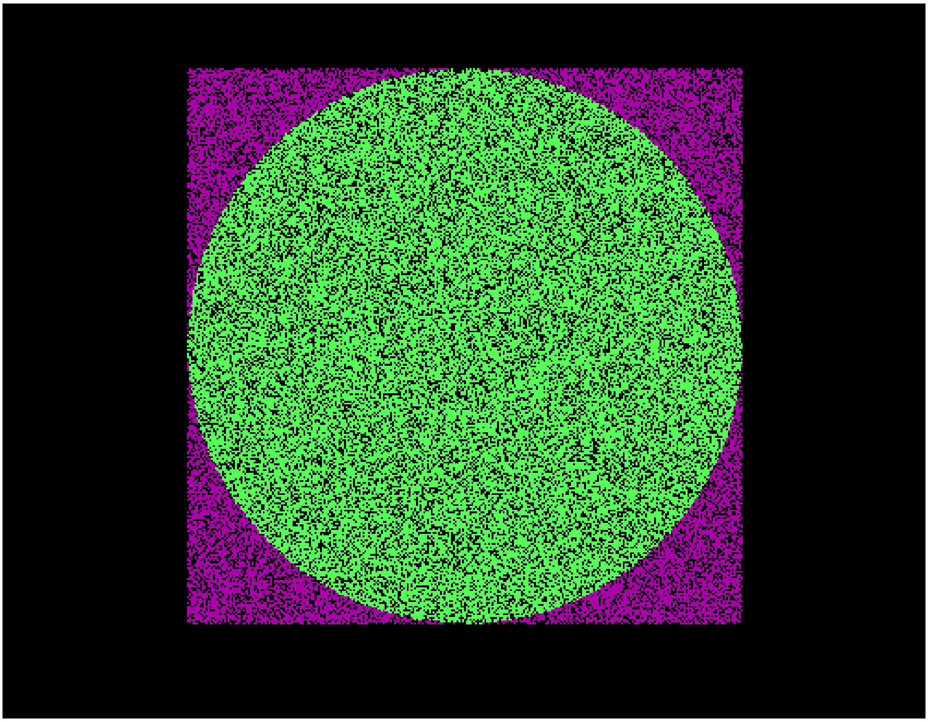

Figure 1: Calculation of π number

3. BUFFON'S NEEDLE PROBLEM

Buffon's Needle Problem; (The naturalist Louis Leclers – Count of Buffon, 1777) is particularly interesting in that π emerges as a result of the solution of a geometric probability problem (Blatner, 2003). This experiment can easily be done by anyone and an estimate for π can be obtained. A plane is separated by parallel lines spaced d units apart. A needle of length l is dropped randomly onto this striped surface. If the needle lands on a line, it is considered a good shot. The probability of a good shot is $2l/d\pi$. When d=1, the probability will be $2/\pi$. If the experiment is done n times and the number of successes is m, $\pi = m/2n$. It is possible to find pages on the Internet that simulate this experiment (See Fig. 2).

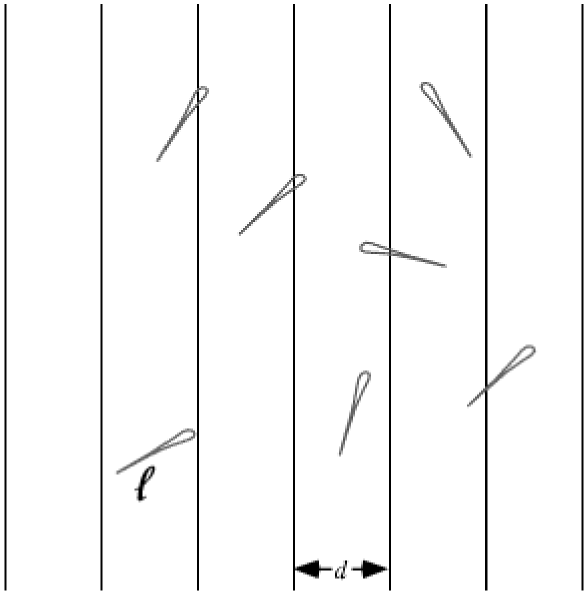

Figure 2: Buffon's Needle Problem;

4. WHAT IS RANDOMNESS?

Theoretical Physicist R. Pagels "What is randomness?" While trying to answer the question, he touched upon the importance of distinguishing between mathematical and physical randomness problems. “A mathematical problem is a logical problem that defines what an arbitrary sequence of numbers or functions means. The physical randomness problem is to determine whether real physical events meet the mathematical criteria for randomness. Until we have a mathematical definition of randomness, we cannot determine whether a sequence of natural events is truly random. “Once we have such a definition, we then have the additional empirical problem of determining whether actual events correspond to such a definition.” (Pagels, 1992; Özbek, 2016)

Let's consider the coin toss experiment and give the value 0 to heads and 1 to heads. Although the probability of occurrence of the sequences 0000000000, 1111111111, 0101010101, 0010100110 are theoretically equal, it is clear that the others, except the fourth sequence, are not random. In the experiment of drawing 10 balls on a return basis from a jar containing balls numbered 0 to 9, respectively; Although the probability of occurrence of the sequences 0123456789 and 0082167489 are theoretically equal, randomness in the first sequence is still suspected.

The definition of a random sequence for arrays whose elements consist of digits of a certain number system is given as follows: When the elements in the array are considered as consecutive k-numbers, if the distribution of these k-numbers in the k-dimensional unit cube is uniform, this sequence is called k-uniform. For example, for k=1, a regular sequence of numbers based on b means that the relative frequency of each digit in the sequence converges to 1/b. If a sequence itself and all its subsequences are k-uniform for every k, this sequence is called a random sequence. 1,0,1,0,1,0,1,0,1,0 sequence 1-smooth, 0,0,0,0,0,0,0,0,0,0 1-not smooth, 1, The sequence 0,1,0,0,1,1,1,0,1 is 1-regular, 2-regular. The fact that the sequence consisting of numbers on the basis of k = 1, b = 10 is 1-order means that the relative frequency of each digit in the sequence converges to 1/10. When it comes to arrays with a finite number of elements in the simulation, how will randomness be ensured for such finite element arrays? Since the series has finite elements, k-smoothness cannot be mentioned at all for large k's (Deak, 1990).

5. STATISTICAL PROPERTIES OF THE DIGITS OF π

Studies to date on the decimal digits of π have shown that these numbers pass all statistical (random) tests (Dodge, 1996; Jaditz, 2000; Lange, 1999; Osler, 1999; Ganz, 2014; Bailey, & Borwein, 2012; Ganz, 2017; Bailey, & Borwein, 2017). It should also be noted that a new statistical test may be developed and these numbers may fail this test. Although it seems that there is no order in these decimal places (which has not been found to date), researchers continue their studies under the assumption that there may be an order. Many methods have been developed and are still being developed to calculate the digits of π. I wonder if there are some statistical features in these steps? We will try to observe these without going into too much statistical information. Let's say that the digits of the number π are in a file (the number of digits we will use is 33,554,400, the programs, π calculator and other documents related to this study can be found at leventmodelleme.com.) I wonder how many of the numbers 0,1,2,…,9 are there. If they are uniformly distributed, they will be found in approximately the same proportions. That is, there will need to be approximately 3,355,440 of each of the numbers 0,1,2,…,9. How many of these numbers there are can be calculated with a short computer program. We can plot the results and see them visually. Accordingly, the number of digits is found as 3355085 3355565 3356623 3355072 3357257 3356378 3353816 3354630 3354113 3355861 and Fig. 3.

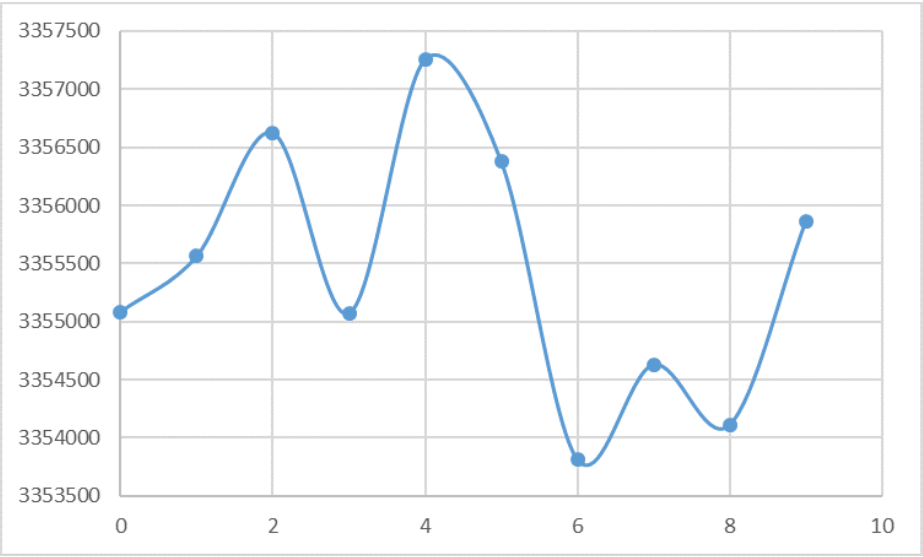

Figure 3: Number of ones

A graph in the form of 1 is obtained. As can be seen from the result, the numbers of the digits 0,1,2,…9 are very close to each other. If these numbers were distributed evenly with a probability of 1/10, 3,355,440 of each would be expected. With these expected values, the Chi-square value can be calculated in Statistics using the observed (counted) values (this is called a goodness-of-fit test of a distribution). When this is calculated, the hypothesis that the observed values are a uniform distribution taking the values 0,1,2,3,4,5,6,7,8,9 with a probability of 1/10 is accepted.

If the numbers in the file are considered side by side, how many of the numbers 00, 01,02,…,99 are there? If a simple computer program is written for this and the results are plotted, a graph in the form of Fig.4 is obtained. When the numbers are considered side by side, there are a total of 33554400/2 = 16,777,200 binaries. We expect there to be 16,777,200/100 = 167,772 numbers 00, 01,…,99. As seen in the figure, the number of pairs is around 167,772. The reason for our expectation here is that the distribution of the numbers 00,01,…99 is a uniform distribution with a probability of 1/100.

Similarly, the Chi-square hypothesis test was performed and it was seen that these had a uniform distribution.

Similarly, while the total number of triplets is 11,184,800 and the expected value for each triplet is 11,184, the number of quadruplets is 8,388,600 and the expected value is 838. Hypothesis testing was conducted for these and it was found that they had a uniform distribution. Thus, 1-2-3-4 uniformity was achieved (see Fig. 4-5). We may wonder whether a larger number of evennesses, for example, 100, 1000, can be achieved statistically. For this, the 33,554,400 digits we have will not be enough; billions to trillions of digits will need to be calculated. Calculating this many digits requires computer hardware and software along with more mathematics.

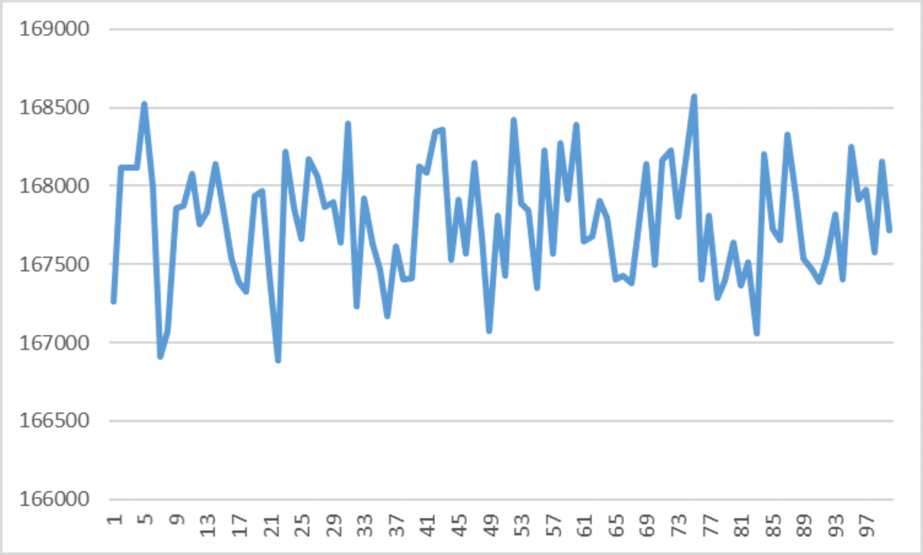

Figure 4: Number of groups of two

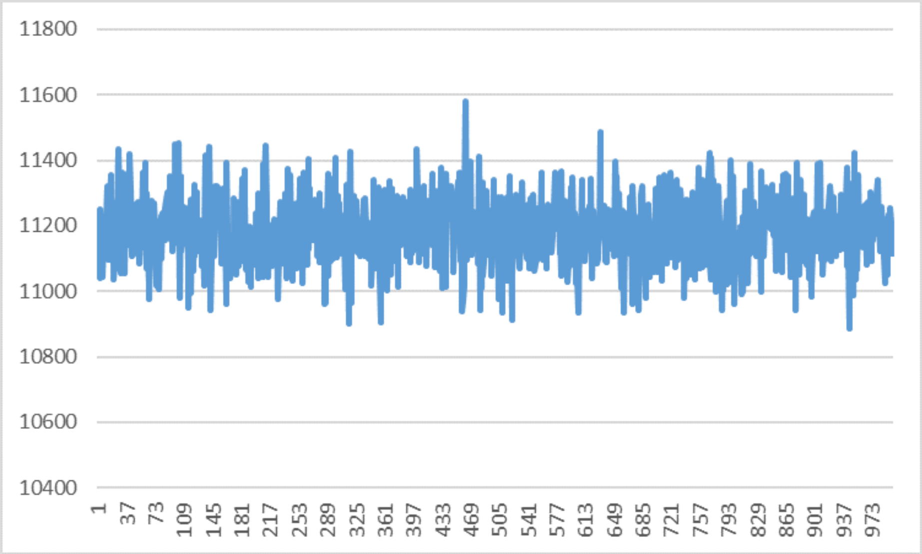

Figure 5: Number of triples

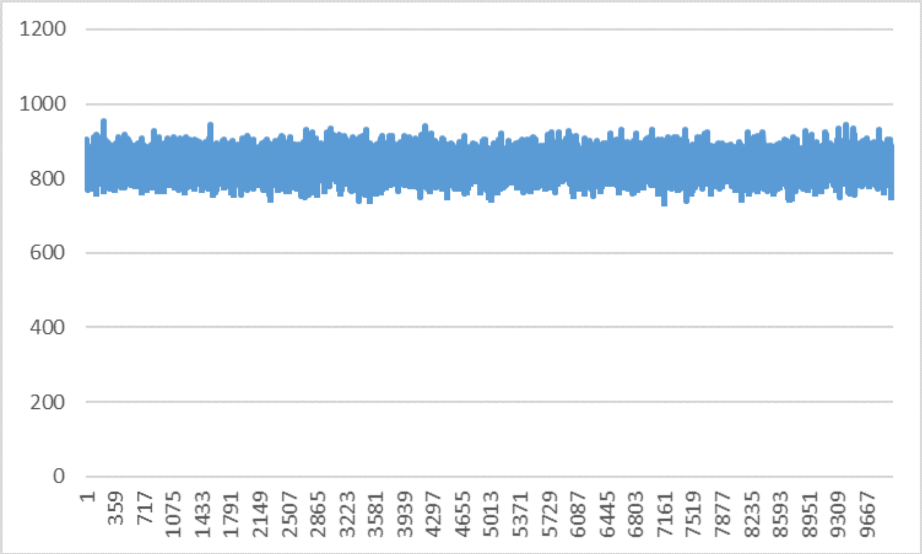

Figure 6: Number of quadruplets

Calculating these steps requires both mathematical knowledge and good algorithm design, and these algorithms are sometimes used to test the speeds of newly developed computers. Billions of digits of are recorded on CDs and in some cases are used as natural random numbers in simulation studies. So far these digits have passed all statistical tests of randomness.

6. SIMULATION STUDY USING DIGITS OF π

Now let's do a simulation study using the digits of π. It can be accomplished as described below without using computer random number generators. Let there be 1000 grasshoppers at a certain starting point and each of them moves 1 unit length in the north-south-east-west directions with equal probability. Let the next grasshopper choose one of these directions and move. Let 1000 of them continue this random movement with equal probability from where they are after moving 1 unit. How can observations be made from this random movement regarding the direction selection of grasshoppers through simulation? We know that the digits of π pass 2 regularity tests. Then, using these steps, a rule can be made for this direction selection. North directions can be selected for 00,01,…,24, south for 25,26,…,49, east for 50,51,…,74, and west for 75,76,…,99. When the grasshoppers jump accordingly and mark the places they go, a beautiful image like the one below emerges. This can be done with a small computer program and can be seen as an animation. If they did not move with equal probability, the digits of π would not be usable. As a result, if grasshoppers are placed in the middle of a field and move randomly with equal probability, they will consume the entire crop of the field after a certain period of time.

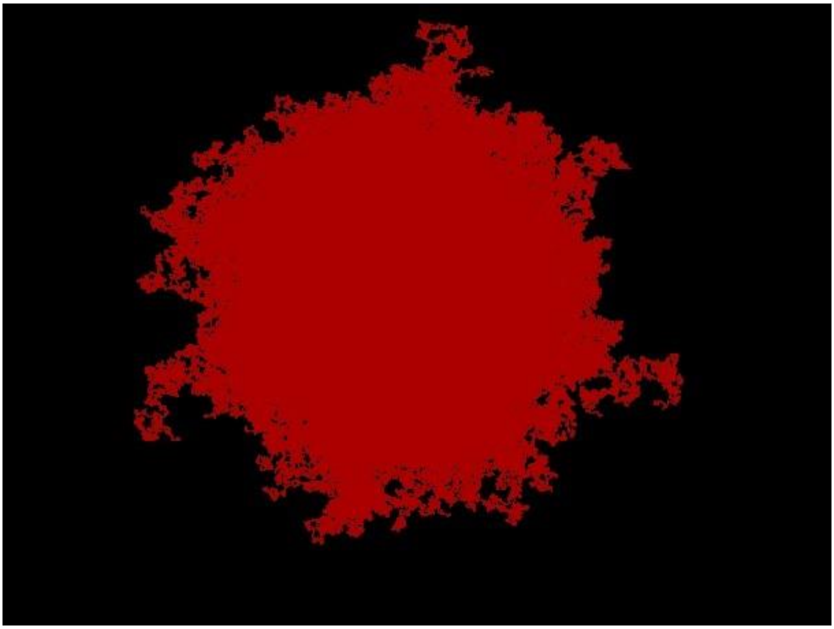

Figure 7: The figure resulting from the simulation

Different ideas can be developed and different research can be conducted to see different results like this. Pictures of the digits of the number π can be drawn and music can be made.

1415926535897932384626433832795028841971693993751058209749445923078164062862089986 28

Let's take the steps again. One step after 1, 1 is encountered, let's call this success. It can be seen as the number of shots made by a basketball player with a probability of success of 1/10 until he achieves the first success. What will be the average of the shots this basketball player makes until he achieves his first success? For example, after this success, he will shoot the basket again after 33 shots, and from now on he will shoot 18 times. It is possible to expand this idea. It is very difficult to count them by eye; by making a simple small computer program, the distribution of the number of shots made until the first success can be revealed. Similar step counts can be made for other digits 0,2,…,9. By using the steps of the number you can obtain from the documents, you can discover various features and create beautiful smiles on your face. This will be an image of dealing with numbers reflected not only in your brain but also in your body.

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Appendix: Digits of π

3.14159265358979323846264338327950288419716939937510582097494459
23078164062862089986280348253421170679821480865132823066470938
44609550582231725359408128481117450284102701938521105559644622
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CHAPTER 2

CONTROLLABILITY OF FRACTIONAL DIFFERENTIAL NEUTRAL SYSTEM HAVING NONPERMUTABLE COEFFICIENTS WITH MULTİPLE DELAYS IN STATE

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INTRODUCTION

Today's real-world problems have stimulated many researchers to generalize and improve the traditional calculus. Although a generalisation of the ordinary calculus in the sense of fractional calculus is based on Leibniz's curiosity(Leibniz, 1962) in 1695, its improvements result from word's necessary needs. Nowadays, fractional calculus is employed in various fields such as diffusion, mathematical physics, signal processing, tomography, mechanism, (Coimbra, 2003; Diethelm & Diethelm, 2010; Heymans & Podlubny, 2006; Kilbas et al., 2006; Obembe et al., 2017; Sweilam & Al-Mekhlafi, 2016; Tarasov, 2019) due to expressing the present issues more appropriate according to the ordinary calculus.

A fractional delay differential equation (Mahmudov & Aydin, 2021; Mahmudov, 2019, 2022; Li & Wang, 2018; You et al.; 2020) is a differential equation consisting of the derivatives of fractional-orders which depend both on the present state variables and on the past state variables. So this constitutes a prerequisite part of social and scientific subjects such as spreads of information or energy and transport in isolated systems. As a special case, a differential equation of integer or fractional orders including the derivatives of integer or fractional orders of the state variables with retardation is known as the neutral fractional(ordinary) differential equations(Pospısil, 2017; Pospisil & Skripkova 2015; You et al., 2021; Zhang et al., 2013) which can be employed from modelling spread of epidemic, population growth to modelling the movements of (radiation) electrons. On looking at the literature, we have observed that many researchers in the studies(Huseynov & Mahmudov, 2022; Pospısil, 2017; Pospisil & Skripkova 2015; You et al., 2021; Zhang et al., 2013) that have considered the neutral fractional(ordinary) differential equations in terms of distinct aspects such as controllability, etc.

The notion of controllability that Kalman firstly put forward at the beginning of 1960s is among the main concepts in contemporary control theory. Even though there exist many distinct definitions about controllability depending on the sorts of control system in the literature, there is no uniform and connected approaches about controllability for nonlinear dynamical control systems due to the lack of general techniques for settling out nonlinear differential equations just as in the situation of linear dynamical systems. Fixed point approach is apparently the strongest technique to acquire the controllability outcomes for nonlinear dynamical control systems(Aydin &

Mahmudov, 2022). In (You, 2020), Zhongli et al. show the controllability of a generalized form of delayed equations of a fractional order with coefficient matrices which do not have to be either zero or permutable investigated by Mahmudov in (Mahmudov, 2019). Pospisil (Pospisil, 2017) examines the controllability of the following neutral equations

$$\begin{gathered}\rho'(\varsigma) - H\rho'(\varsigma-\tau) = A\rho(\varsigma-\tau) + g(\varsigma), \quad \varsigma > 0,\ \tau > 0, \\ \rho(\varsigma) = \psi(\varsigma)\ ,\ -\tau \le \varsigma \le 0,\end{gathered} \tag{1}$$

Pospisil (Pospisil, 2017) achieves to define each control function of system (1) by means of the shifted Legendre functions and provided the same Kalman type condition for the controllability of the just-given equations (1). Researchers in the study (You et al., 2021) prove the controllability for the following equations (2) with permutable matrices via the fixed point theorem of the Krasnoselskii

$$\begin{gathered}\rho'(\varsigma) - H\rho'(\varsigma-\tau) = Z\rho(\varsigma) + A\rho(\varsigma-\tau) + g(\varsigma), \quad \varsigma > 0, \tau > 0, \\ \rho(\varsigma) = \psi(\varsigma)\ ,\ -\tau \le \varsigma \le 0,\end{gathered} \tag{2}$$

Lastly, Aydin and Mahmudov (Aydin & Mahmudov, 2022) were able to demonstrate relative controllability for the below differential neutral multi-delayed system consisting of the classical Caputo fractional derivatives with non-permutable coefficient matrices

$$\begin{gathered}{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\left[\rho(\varsigma) - \sum_{k=1}^{d} H_k\rho(\varsigma-\tau_k)\right] = Z\rho(\varsigma) + \sum_{k=1}^{d} A_k\rho(\varsigma-\tau_k) + g(\varsigma) \\ \rho(\varsigma) = \psi(\varsigma), -\tau \le \varsigma \le 0,\end{gathered} \tag{3}$$

Aydin and Mahmudov (Aydin & Mahmudov, 2022) have made analysis of existence uniqueness and stability for the following differential multi-delayed neutral system consisting of the Caputo derivative w.r.t another function ϑ with nonpermutable matrices

$${}_{0^+}^{C}\beth_{\vartheta}^{\alpha}\left[\rho(\varsigma) - \sum_{k=1}^{d} H_k\rho(\varsigma-\tau_k)\right] = Z\rho(\varsigma) + \sum_{k=1}^{d} A_k\rho(\varsigma-\tau_k) + g(\varsigma), \tag{4}$$

$$\rho(\varsigma) = \psi(\varsigma), -\tau \leq \varsigma \leq 0,$$

where ${}_{0^+}^{C}\beth_{\vartheta}^{\alpha}$ is the Caputo derivative w.r.t another function ϑ is of fractional order $0 < \alpha < 1$. $H_k, Z, A_k, k = 1,2,3, \ldots, d$ are non-permutable n-by-n constant coefficient matrices $\tau_k > 0$ is a delay for each of $k = 1,2,3, \ldots, d$ and $\tau = \max\{\tau_k : k = 1,2,3, \ldots, d\}$. $g \in C([0,T] \times \mathbb{R}^n, \mathbb{R}^n)$, $\psi \in C^1([0,T], \mathbb{R}^n)$ with $T = ld$, a fixed natural number l. Furthermore, I will relatively handle the controllability of ${\vartheta}$-fractional differential neutral multi-delayed equation having non-permutable coefficients (4) via ϑ-multi-delayed perturbation of M-L functions by taking inspiration from the studies (Aydin & Mahmudov, 2022; Muthuvel, 2023; You et al., 2020).

In the present work,

- A notion of the ϑ-Gramian matrix is given.
- The sufficient and necessary circumstances are determined so as to relatively demonstrate that the ϑ-Caputo fractional linear multi-delayed neutral equation is controllable.
- The controllability outcomes of the ϑ-Caputo fractional semi-linear multi-delayed neutral system via the fixed point theorem of the Krasnoselskii.
- The theoretical outcomes are illustrated.

1. PRELIMINARIES

With n, m being the natural numbers, $C^m([0,T], \mathbb{R}^n)$ consists of such functions that they have continuous derivatives up to order $m - 1$. For $m = 1$, it is endowed with the norm

$$\|\rho\|_C = \sup_{\varsigma \in [0,T]} \|\rho(\varsigma)\|$$

for an arbitrary norm $\|.\|$ on $\mathbb{R}^n$. $AC([0,T], \mathbb{R}^n)$ consists of all functions which are absolutely continuous on $[0,T]$. $AC^m([0,T], \mathbb{R}^n)$ consists of such functions that they have absolutely continuous derivatives up to order $m - 1$. A norm for an arbitrary matrix $B \in \mathbb{R}^{m \times m}$

$$\|B\| = \max_{1 \leq i \leq m} \sum_{k=1}^{m} |b_{ik}|$$

here, b_{ik} stands for the entries of B. With Z_1, Z_2 being Banach spaces, a Banach space $B(Z_1, Z_2)$ is consists of such linear operators that it is bounded from Z_1 to Z_2. $L^{\infty}(J, Z_2)$ is Banach with $\|.\|_{L^{\infty}(J,Z_2)}$ for J being an arbitrary closed bounded interval. Let such a continuously differentiable function ϑ on $[0, T]$ that it is increasing and $\vartheta'(\varsigma) \neq 0, 0 \leq \varsigma \leq T$. In (Almeida, 2017; Kilbas et al., 2006), R-L integral w.r.t another function ϑ of $\zeta \in AC^n[0, T]$ of the fractional order $\alpha > 0$ is described as follows

$$\left({}^{RL}_{0^+}\gimel^{\alpha}_{\vartheta}\zeta\right)(\varsigma) := \frac{1}{\Gamma(n-\alpha)} \int_0^{\varsigma} \left(\vartheta(\varsigma) - \vartheta(s)\right)^{\alpha-1} \zeta(s) d\vartheta(s).$$

R-L fractional derivatives w.r.t another function ϑ of $\zeta \in AC^n[0, T]$ of order $\alpha > 0$ are defined as follows

$$\left({}^{RL}_{0^+}\beth^{\alpha}_{\vartheta}\zeta\right)(\varsigma) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{d\vartheta(\varsigma)}\right)^n \int_0^{\varsigma} \left(\vartheta(\varsigma) - \vartheta(s)\right)^{n-\alpha-1} \zeta(s) d\vartheta(s),$$

where $n = [\alpha] + 1$. Let $\zeta \in AC^n([0, T], \mathbb{R})$ and let $\vartheta \in C^n([0, T], \mathbb{R}^n)$ be increasing with $\vartheta'(\varsigma) \neq 0$ for $\varsigma \in [0, T]$, then the Caputo derivatives w.r.t another function ϑ of ζ of the fractional order α is described as noted below

$${}^{C}_{0^+}\beth^{\alpha}_{\vartheta}\zeta(\varsigma) := {}^{RL}_{0^+}\gimel^{n-\alpha}_{\vartheta} \left(\frac{d}{d\vartheta(\varsigma)}\right)^n \zeta(\varsigma).$$

Defnition 1.1 (Aydin & Mahmudov, 2022a) System (4) is said to be relatively controllable, when there exists such a square integrable control v that equations (4) is of $\rho \in C^1([-\tau, \varsigma], \mathbb{R}^n)$ that satisfies the initial $\rho(\varsigma) = \psi(\varsigma)$ for $\varsigma \in [-\tau, 0]$ and $\rho(\sigma) = \rho_{\sigma}$, where an arbitrary final state $\rho(\sigma) \in \mathbb{R}^n$ with an arbitrary time σ, and an arbitrary initial function $\psi \in C^1([-\tau, 0], \mathbb{R}^n)$.

Theorem 1.1 (Aydin & Mahmudov, 2022b) The continuous solution to the system (4) can be offered by

$$\rho(\varsigma) = \left[\mathcal{X}^{H,Z,A}_{\alpha,\alpha,\vartheta}(\varsigma, 0) - \sum_{i=1}^{d} H_i \mathcal{X}^{H,Z,A}_{\alpha,1,\vartheta}(\varsigma, \tau_i)\right] \psi(0) + \int_0^{\varsigma} \mathcal{X}^{H,Z,A}_{\alpha,\alpha,\vartheta}(\varsigma, s) g\left(s, \rho(s)\right) d\vartheta(s)$$

$$+ \sum_{i=1}^{d} \int_{-\tau_i}^{0} \mathcal{X}^{H,Z,A}_{\alpha,1,\vartheta}(\varsigma, s + \tau_i) \left[H_i \left({}^{C}_{0^+}\beth^{\alpha}_{\vartheta}\psi\right)(s) + A_i \psi(s)\right] d\vartheta(s)$$

where the ϑ-multi-delayed perturbation(MDP) of the M-L function

$$\mathcal{X}^{H,Z,A}_{\alpha,\beta,\vartheta}(\varsigma,s)=\begin{cases}\Theta, & -\tau\le\varsigma<0,\ s\ge 0,\\ \sum_{k=0}^{\infty}\sum_{i_1,i_2,\ldots,i_d=0}^{\infty}\mathcal{Q}_{k+1}\dfrac{\left[\vartheta(\varsigma)-\vartheta\left(s+\sum_{j=1}^{d}i_j\tau_j\right)\right]_+^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)}, & \varsigma\ge 0, s\ge 0,\end{cases} \tag{5}$$

and $\mathcal{Q}_{k+1}:=Q_{k+1}(i_1\tau_1,\ldots,i_d,\tau_d)$, $A=\sum_{k=1}^{d}A_k$, $H=\sum_{k=1}^{d}H_k$, $[\varsigma]_+=\max(0,\varsigma)$, and

$$Q_{j+1}(s_1,s_2,\ldots,s_d)=BQ_j(s_1,s_2,\ldots,s_d)+\sum_{k=1}^{d}F_k\,Q_j(s_1,s_2,\ldots,s_k-\tau_k,\ldots,s_d)$$

$$+\sum_{k=1}^{d}A_k\,Q_{j+1}(s_1,s_2,\ldots,s_k-\tau_k,\ldots,s_d),$$

$$Q_j(s_1,\ldots,-\tau_d)=\Theta=Q_j(-\tau_d,\ldots,s_1)=Q_0(s_1,s_2,\ldots,s_d),$$

$$Q_1(0,\ldots,0)=I,\qquad s_i\ne 0, Q_1(s_1,s_2,\ldots,s_d)=\Theta$$

where I and Θ are the unit and zero matrices, respectively.

2. PRINCIPLE CONTRIBUTIONS

From here on, we offer our main contributions.

Theorem 2.1 The ϑ-MDP function $\mathcal{X}^{H,Z,A}_{\alpha,\beta,\vartheta}(\varsigma,s)$ is continuous in $0<s<\varsigma<\infty$.

Proof. For an arbitrary $\varsigma_0>0$ and a fixed $s_0>0$, we consider

$$\lim_{\varsigma\to\varsigma_0} \mathcal{X}_{\alpha,\beta,\vartheta}^{H,Z,A}(\varsigma, s_0) = \lim_{\varsigma\to\varsigma_0} \sum_{k=0}^{\infty} \sum_{i_1,i_2,\dots,i_d=0}^{\infty} \mathcal{Q}_{k+1} \frac{\left[\vartheta(\varsigma) - \vartheta\left(s_0 + \sum_{j=1}^{d} i_j\tau_j\right)\right]_+^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)}$$

$$= \sum_{k=0}^{\infty} \sum_{i_1,i_2,\dots,i_d=0}^{\infty} \mathcal{Q}_{k+1} \lim_{\varsigma\to\varsigma_0} \frac{\left[\vartheta(t) - \vartheta\left(s_0 + \sum_{j=1}^{d} i_j\tau_j\right)\right]_+^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)}$$

$$= \sum_{k=0}^{\infty} \sum_{i_1,i_2,\dots,i_d=0}^{\infty} \mathcal{Q}_{k+1} \frac{\left[\vartheta(\varsigma_0) - \vartheta\left(s_0 + \sum_{j=1}^{d} i_j\tau_j\right)\right]_+^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)}$$

$$= \mathcal{X}_{\alpha,\beta,\vartheta}^{H,Z,A}(\varsigma_0, s_0).$$

Lemma 2.1. The function $\mathcal{X}_{\alpha,\beta,\vartheta}^{H,Z,A}$ which is defined as in (5) satisfies the following expressions.

$$\left(\mathcal{X}_{\alpha,\beta,\vartheta}^{H,Z,A}\right)^T(\varsigma, s) = \mathcal{X}_{\alpha,\beta,\vartheta}^{H^T,Z^T,A^T}(\varsigma, s)$$

and

$$\int_0^{\varsigma} \mathcal{X}_{\alpha,\beta,\vartheta}^{H,Z,A}(\varsigma, s)\, d\vartheta(s) \le \left(\vartheta(\varsigma) - \vartheta(0)\right)\mathcal{X}_{\alpha,\beta,\vartheta}^{H,Z,A}(\varsigma, 0).$$

where $.^T$ stands for the transpose of an arbitrary matrix.

Proof. We firstly consider the first item as follows

$$\left(\mathcal{X}_{\alpha,\beta,\vartheta}^{H,Z,A}\right)^T(\varsigma, s) = \left[\sum_{k=0}^{\infty} \sum_{i_1,i_2,\dots,i_d=0}^{\infty} \mathcal{Q}_{k+1} \frac{\left[\vartheta(\varsigma) - \vartheta\left(s + \sum_{j=1}^{d} i_j\tau_j\right)\right]_+^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)}\right]^T$$

$$= \sum_{k=0}^{\infty} \sum_{i_1,i_2,\dots,i_d=0}^{\infty} \mathcal{Q}_{k+1}^T \frac{\left[\vartheta(\varsigma) - \vartheta\left(s + \sum_{j=1}^{d} i_j\tau_j\right)\right]_+^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)}$$

$$:= \mathcal{X}_{\alpha,\beta,\vartheta}^{H^T,Z^T,A^T}(\varsigma, s),$$

and we secondly take the second item into consideration

$$\int_0^{\varsigma} \mathcal{X}_{\alpha,\beta,\vartheta}^{H,Z,A}(\varsigma,s)\,d\vartheta(s) = \int_0^{\varsigma} \sum_{k=0}^{\infty} \sum_{i_1,i_2,\dots,i_d=0}^{\infty} \mathcal{Q}_{k+1} \frac{\left[\vartheta(\varsigma) - \vartheta\left(s + \sum_{j=1}^{d} i_j\tau_j\right)\right]_+^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)} d\vartheta(s)$$

$$\leq \sum_{k=0}^{\infty} \sum_{i_1,i_2,\dots,i_d=0}^{\infty} \mathcal{Q}_{k+1} \frac{\left[\vartheta(\varsigma) - \vartheta\left(s + \sum_{j=1}^{d} i_j\tau_j\right)\right]_+^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)} \int_0^{\varsigma} d\vartheta(s)$$

$$= \left(\vartheta(\varsigma) - \vartheta(0)\right)\mathcal{X}_{\alpha,\beta,\vartheta}^{H,Z,A}(\varsigma,0).$$

Lemma 2.2 If $\alpha \in (0,1)$ and $\beta \in (0,1]$ satisfies $\alpha+\beta > 1$, then the below inequality holds true

$$\left\|\mathcal{X}_{\alpha,\beta,\vartheta}^{H,Z,A}(\varsigma,s)\right\| \leq \mathcal{X}_{\alpha,\beta,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\varsigma,s),$$

for all $\varsigma \geq 0$, a fixed $s \geq 0$.

Proof. By implementing the triangle inequality, one acquires

$$\left\|\mathcal{X}_{\alpha,\beta,\vartheta}^{H,Z,A}(\varsigma,s)\right\| = \left\|\sum_{k=0}^{\infty} \sum_{i_1,i_2,\dots,i_d=0}^{\infty} \mathcal{Q}_{k+1} \frac{\left[\vartheta(\varsigma) - \vartheta\left(s + \sum_{j=1}^{d} i_j\tau_j\right)\right]_+^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)}\right\|$$

$$\leq \sum_{k=0}^{\infty} \sum_{i_1,i_2,\dots,i_d=0}^{\infty} \|\mathcal{Q}_{k+1}\| \frac{\left[\vartheta(\varsigma) - \vartheta\left(s + \sum_{j=1}^{d} i_j\tau_j\right)\right]_+^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)}$$

$$:= \mathcal{X}_{\alpha,\beta,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\varsigma,s)$$

We have two cases to investigate, the first case: $g\left(\varsigma,\rho(\varsigma)\right) = 0, \varsigma \in [0,\sigma] = J$, that is,

$$ {}_{0^+}^{C}\beth_{\vartheta}^{\alpha}\left[\rho(\varsigma) - \sum_{j=1}^{d} H_j\rho\left(\varsigma - \tau_j\right)\right] = Z\rho(\varsigma) + \sum_{j=1}^{d} A_j\rho\left(\varsigma - \tau_j\right) + Sv(\varsigma), \varsigma > 0, \tag{6}$$

$$\rho(\varsigma) = \psi(\varsigma), -\tau \leq \varsigma \leq 0,$$

whose solution is

$$\rho(\varsigma) = \left[\mathcal{X}_{\alpha,1,\vartheta}^{H,Z,A}(\varsigma,0) - \sum_{m=1}^{d} H_m\,\mathcal{X}_{\alpha,1,\vartheta}^{H,Z,A}(\varsigma,\tau_m)\right]\psi(0) + \int_0^{\varsigma} \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma,x)\,Sv(x)d\vartheta(x)$$

$$+ \sum_{m=1}^{d} \int_{-\tau_m}^{0} \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma,x+\tau_m)\left[H_m\left({}_{0^+}^{C}\beth_{\vartheta}^{\alpha}\psi\right)(x) + A_m\psi(x)\right]d\vartheta(x)$$

Note that for simplicity, we use $\vartheta_u^v = \vartheta(v) - \vartheta(u),\ u, v \in J$. So, we have $\vartheta_0^\varsigma = \vartheta(\varsigma) - \vartheta(u_0)$. Then

$$\int_0^\varsigma \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma, s)\, d\vartheta(s) \leq\ \vartheta_0^\varsigma \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma, 0).$$

Now, we will offer a representation of ϑ-multi-delayed Gramian matrix below

$$\mathfrak{M}_{\tau,\alpha,\vartheta}[0,\sigma] = \int_0^\varsigma \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma, s)\, SS^T \mathcal{X}_{\alpha,\alpha,\vartheta}^{H^T,Z^T,A^T}(\sigma, s) d\vartheta(s)$$

where $H^T = \sum_{n=1}^d H_n^T$ and $A^T = \sum_{n=1}^d A_n^T$.

Theorem 2.2 $\mathfrak{M}_{\tau,\alpha,\vartheta}[0,\sigma]$is invertible if and only if the system (6) is relatively controllable .

Proof. Sufficiency: Assume that $\mathfrak{M}_{\tau,\alpha,\vartheta}[0,\sigma]$ is not invertible. Then if we see columns of $\mathfrak{M}_{\tau,\alpha,\vartheta}[0,\sigma]$ as vectors, then they are dependent and so there is at least such a coeffcient vector $b \in \mathbb{R}^n$ that

$$\mathfrak{M}_{\tau,\alpha,\vartheta}[0,\sigma] b = 0.$$

One acquires

$$0 = b^T \mathfrak{M}_{\tau,\alpha,\vartheta}[0,\sigma] b = b^T \int_0^\sigma \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma, s)\, SS^T \mathcal{X}_{\alpha,\alpha,\vartheta}^{H^T,Z^T,A^T}(\sigma, s) d\vartheta b$$
$$= \int_0^\sigma \left\| b^T \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma, s) S \right\|^2 d\vartheta(s),$$

which shows that

$$b^T \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma, s) S = 0, \quad 0 \leq s \leq \sigma.$$

Because of the relative controllability of system (6), there exist control functions $v_1, v_2 \in L^2(J, \mathbb{R}^n)$ for the final function states $0, b \in \mathbb{R}^n$ with time σ, such that system (6) is of a solution $\rho \in C^1([-\tau, \sigma], \mathbb{R}^n)$ satisfying $\rho(\sigma) = 0$ and $\rho(\sigma) = b$ together with the initial ψ, i.e.,

$$\rho(\sigma) = \left[\mathcal{X}_{\alpha,1,\vartheta}^{H,Z,A}(\sigma, 0) - \sum_{m=1}^{d} H_m \mathcal{X}_{\alpha,1,\vartheta}^{H,Z,A}(\sigma, \tau_m) \right] \psi(0)$$

$$+\sum_{m=1}^{d}\int_{-\tau_m}^{0}\chi_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x+\tau_m)\left[H_m\left({}_{0^+}^{C}\beth_{\vartheta}^{\alpha}\psi\right)(x)+A_m\psi(x)\right]d\vartheta(x)$$

$$+\int_0^{\sigma}\chi_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x)\,Sv_1(x)d\vartheta(x)=0$$

and

$$\rho(\sigma)=\left[\chi_{\alpha,1,\vartheta}^{H,Z,A}(\sigma,0)-\sum_{m=1}^{d}H_m\,\chi_{\alpha,1,\vartheta}^{H,Z,A}(\sigma,\tau_m)\right]\psi(0)$$

$$+\sum_{m=1}^{d}\int_{-\tau_m}^{0}\chi_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x+\tau_m)\left[H_m\left({}_{0^+}^{C}\beth_{\vartheta}^{\alpha}\psi\right)(x)+A_m\psi(x)\right]d\vartheta(x)$$

$$+\int_0^{\sigma}\chi_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x)\,Sv_2(x)d\vartheta(x)=b$$

One can easily acquire

$$b=\int_0^{\sigma}\chi_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x)\,S[v_2(x)-v_1(x)]d\vartheta(x)$$

one obtains the following from the just-above equality by multiplying byg^T,

$$b^Tb=\int_0^{\sigma}b^T\chi_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x)\,S[v_2(x)-v_1(x)]d\vartheta(x)$$

This together with (8) shows $b=0$. It is impossible because g was a nonzero.

Necessity: Because $\mathfrak{M}_{\tau,\alpha,\vartheta}[0,\sigma]$is invertible, then $\left(\mathfrak{M}_{\tau,\alpha,\vartheta}[0,\sigma]\right)^{-1}$exists. The

just-below control function for an arbitrary final state g can be chosen

$$v(s)=\left(S^T\chi_{\alpha,\alpha,\vartheta}^{H^T,Z^T,A^T}(\sigma,s)\right)\left(\mathfrak{M}_{\tau,\alpha,\vartheta}[0,\sigma]\right)^{-1}\eta$$

where

$$\eta=b-\left[\chi_{\alpha,1,\vartheta}^{H,Z,A}(\sigma,0)-\sum_{m=1}^{d}H_m\,\chi_{\alpha,1,\vartheta}^{H,Z,A}(\sigma,\tau_m)\right]\psi(0)$$

$$-\sum_{m=1}^{d}\int_{-\tau_m}^{0} \chi_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x+\tau_m)\left[H_m\left({}_{0^+}^{C}\beth_{\vartheta}^{\alpha}\psi\right)(x)+A_m\psi(x)\right]d\vartheta(x).$$

Then,

$$\begin{aligned}
\rho(\sigma) = &\left[\chi_{\alpha,1,\vartheta}^{H,Z,A}(\sigma,0)-\sum_{m=1}^{d} H_m\,\chi_{\alpha,1,\vartheta}^{H,Z,A}(\sigma,\tau_m)\right]\psi(0)\\
&+\sum_{m=1}^{d}\int_{-\tau_m}^{0} \chi_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x+\tau_m)\left[H_m\left({}_{0^+}^{C}\beth_{\vartheta}^{\alpha}\psi\right)(x)+A_m\psi(x)\right]d\vartheta(x)\\
&+\int_0^{\sigma} \chi_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x)\,Sv(x)d\vartheta(x)\\
&=\left[\chi_{\alpha,1,\vartheta}^{H,Z,A}(\sigma,0)-\sum_{m=1}^{d} H_m\,\chi_{\alpha,1,\vartheta}^{H,Z,A}(\sigma,\tau_m)\right]\psi(0)\\
&+\sum_{m=1}^{d}\int_{-\tau_m}^{0} \chi_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x+\tau_m)\left[H_m\left({}_{0^+}^{C}\beth_{\vartheta}^{\alpha}\psi\right)(x)+A_m\psi(x)\right]d\vartheta(x)\\
&+\int_0^{\sigma} \chi_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x)\,S\left(S^T\chi_{\alpha,\alpha,\vartheta}^{H^T,Z^T,A^T}(\sigma,s)\right)\left(\mathfrak{M}_{\tau,\alpha,\vartheta}[0,\sigma]\right)^{-1}\eta d\vartheta(x)
\end{aligned}$$

$$\begin{aligned}
&=\left[\chi_{\alpha,1,\vartheta}^{H,Z,A}(\sigma,0)-\sum_{m=1}^{d} H_m\,\chi_{\alpha,1,\vartheta}^{H,Z,A}(\sigma,\tau_m)\right]\psi(0)\\
&+\sum_{m=1}^{d}\int_{-\tau_m}^{0} \chi_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x+\tau_m)\left[H_m\left({}_{0^+}^{C}\beth_{\vartheta}^{\alpha}\psi\right)(x)+A_m\psi(x)\right]d\vartheta(x)+\eta\\
&=b.
\end{aligned}$$

Now we investigate the second case:$g(\varsigma,\rho(\varsigma))\neq\mathbb{R}^n$, $\varsigma\in J=[0,\sigma]$, that is,

$$\begin{gathered}
{}_{0^+}^{C}\beth_{\vartheta}^{\alpha}\left[\rho(\varsigma)-\sum_{k=1}^{d} H_k\rho(\varsigma-\tau_k)\right]=Z\rho(\varsigma)+\sum_{k=1}^{d} A_k\rho(\varsigma-\tau_k)+Sv(\varsigma)+g(\varsigma,\rho(\varsigma)), \qquad (9)\\
\rho(\varsigma)=\psi(\varsigma),-\tau\leq\varsigma\leq 0,
\end{gathered}$$

with the solution of a form

$$\rho(\varsigma) = \left[\mathcal{X}_{\alpha,1,\vartheta}^{H,Z,A}(\varsigma,0) - \sum_{m=1}^{d} H_m\, \mathcal{X}_{\alpha,1,\vartheta}^{H,Z,A}(\varsigma,\tau_m)\right]\psi(0) + \int_0^{\varsigma} \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma,x)\, Sv(x)d\vartheta(x)$$
$$+ \sum_{m=1}^{d}\int_{-\tau_m}^{0} \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma,x+\tau_m)\left[H_m\left({}_{0^+}^{C}\beth_{\vartheta}^{\alpha}\psi\right)(x) + A_m\psi(x)\right]d\vartheta(x) \tag{10}$$
$$+ \int_0^{\varsigma} \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma,x)\, g\big(\mathrm{x},\rho(x)\big)d\vartheta(x).$$

Now, we need extra assumptions

$(\boldsymbol{O_1})$ $\mathfrak{M}_{\vartheta,c}$ is an operatör from $L^2(J\times\mathbb{R}^n)$ to $\mathbb{R}^n$ offered by

$$\mathfrak{M}_{\vartheta,c}v = \int_0^{\sigma} \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x)\, Sv(x)d\vartheta(x),$$

has an inverse operator $\left(\mathfrak{M}_{\vartheta,c}\right)^{-1}$ taking values from $L^2(J\times\mathbb{R}^n)/\,ker\mathfrak{M}_{\vartheta,c}$.

$(\boldsymbol{O_2})$ g is continuous from $J\times\mathbb{R}^n$ to $\mathbb{R}^n$ and $L_g(.)\in L^{\infty}(J,\mathbb{R}^n)$ such that for $\rho, w\in\mathbb{R}^n$

$$\left\|g\big(\varsigma,\rho(\varsigma)\big) - g\big(\varsigma,w(\varsigma)\big)\right\| \le L_g(\varsigma)\|\rho(\varsigma)-w(\varsigma)\|, \varsigma\in J.$$

Setting:

$$R = \left\|\mathfrak{M}_{\vartheta,c}\right\|_{B\left(\mathbb{R}^n,\, L^2(J\times\mathbb{R}^n)/\,ker\mathfrak{M}_{\vartheta,c}\right)}^{-1},$$

$$R_1 = \mathcal{X}_{\alpha,1,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,0)\|\psi(0)\| + \sum_{m=1}^{d}\|H_m\|\, \mathcal{X}_{\alpha,1,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,\tau_m)\|\psi(0)\|$$
$$+ \sum_{m=1}^{d}\int_{-\tau_m}^{0} \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,x+\tau_m)\left\|H_m\left({}_{0^+}^{C}\beth_{\vartheta}^{\alpha}\psi\right)(x) + A_m\psi(x)\right\|d\vartheta(x)$$
$$+N_g\vartheta_0^{\sigma}\mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,0),$$

and

$$R_2 = \vartheta_0^{\sigma}\mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,0)\left\|L_g\right\|_{L^{\infty}(J,\mathbb{R}^n)}$$

where $N_g = max_{\varsigma\in[0,\sigma]}\|g(\varsigma,0)\|$. The below information follows from (Wang et al., 2017) that

$$R = \sqrt{\left\|\mathfrak{M}_{\tau,\alpha,\vartheta}[0,\sigma]\right\|}.$$

Theorem 2.3 Let $\alpha \in [0.5,1)$. $(\boldsymbol{O_1})$ and $(\boldsymbol{O_2})$ are satisfied. Equations (9) are relatively controllable provided that

$$R_2\left(1+\vartheta_0^{\sigma}\mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,0)\|S\|R\right)<1. \tag{11}$$

Proof. Based on $(\boldsymbol{O_1})$ the following control operator $v_\rho(\varsigma)$ can be defined by

$$\begin{aligned} v_\rho(\varsigma) = \left(\mathfrak{M}_{\vartheta,c}\right)^{-1}\Bigg[\rho_\sigma - \mathcal{X}_{\alpha,1,\vartheta}^{H,Z,A}(\sigma,0)\psi(0) - \sum_{m=1}^{d} H_m\,\mathcal{X}_{\alpha,1,\vartheta}^{H,Z,A}(\sigma,\tau_m)\psi(0) \\ - \sum_{m=1}^{d}\int_{-\tau_m}^{0}\mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x+\tau_m)\left[H_m\left({}_{0^+}^{C}\beth_{\vartheta}^{\alpha}\psi\right)(x)+A_m\psi(x)\right]d\vartheta(x) \\ - \int_0^{\sigma}\mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x)\,g\big(\mathrm{x},\rho(x)\big)d\vartheta(x)\Bigg](\varsigma). \end{aligned} \tag{12}$$

By using the control operator, one can describe $\mathcal{K}: C = C(J,\mathbb{R}^n) \to C$ by

$$\begin{aligned} \mathcal{K}\rho(\varsigma) = \left[\mathcal{X}_{\alpha,1,\vartheta}^{H,Z,A}(\varsigma,0) - \sum_{m=1}^{d} H_m\,\mathcal{X}_{\alpha,1,\vartheta}^{H,Z,A}(\varsigma,\tau_m)\right]\psi(0) + \int_0^{\varsigma}\mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma,x)\,Sv_\rho(x)d\vartheta(x) \\ + \sum_{m=1}^{d}\int_{-\tau_m}^{0}\mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma,x+\tau_m)\left[H_m\left({}_{0^+}^{C}\beth_{\vartheta}^{\alpha}\psi\right)(x)+A_m\psi(x)\right]d\vartheta(x) \\ + \int_0^{\varsigma}\mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma,x)\,g\big(\mathrm{x},\rho(x)\big)d\vartheta(x). \end{aligned} \tag{13}$$

which has a fixed point ρ satisfying system (9). When keeping in mind the definition of relative controllability, equations (9) with (12) is relatively controllable iff (13) has a solution $\rho \in C([-\tau,\sigma],\mathbb{R}^n)$ so that $\rho(\sigma) = \rho_\sigma$ and $\rho(\varsigma) = \psi(\varsigma)$, $\varsigma \in [-\tau,0]$. It is well-known that for each $\varepsilon > 0$, $\mathcal{D}_\varepsilon = \{\rho \in C: \|\rho\|_C < \varepsilon\}$ is a both convex and bounded set which is closed. The rest of the proof is divided into three stages so as to get it understandable.

<u>*Step 1: We*</u> are going to determine at least $\varepsilon > 0$ such that

$$\mathcal{K}(\mathcal{D}_\varepsilon) \subseteq \mathcal{D}_\varepsilon.$$

In the light of $(\boldsymbol{O_1})$ and $(\boldsymbol{O_2})$ and Lemma 2.1 and Hölder inequality, we get the following inequality for the norm of the control $v_\rho(\varsigma)$

$$\|v_\rho(\varsigma)\| = \left\|\left(\mathfrak{M}_{\vartheta,c}\right)^{-1}\right\|\left[\|\rho_\sigma\| + \left\|\mathcal{X}^{H,Z,A}_{\alpha,1,\vartheta}(\sigma,0)\right\|\|\psi(0)\| + \left\|\sum_{m=1}^{d} H_m\,\mathcal{X}^{H,Z,A}_{\alpha,1,\vartheta}(\sigma,\tau_m)\right\|\|\psi(0)\|\right.$$

$$+\sum_{m=1}^{d}\int_{-\tau_m}^{0}\left\|\mathcal{X}^{H,Z,A}_{\alpha,\alpha,\vartheta}(\sigma,x+\tau_m)\right\|\left[\left\|H_m\left({}_{0^+}^{C}\beth^{\alpha}_{\vartheta}\psi\right)(x) + A_m\psi(x)\right\|\right]d\vartheta(x)$$

$$\left.+\int_0^\sigma\left\|\mathcal{X}^{H,Z,A}_{\alpha,\alpha,\vartheta}(\sigma,x)\right\|\left\|g\big(\mathrm{x},\rho(x)\big)\right\|d\vartheta(x)\right]$$

$$\le R\left[\|\rho_\sigma\| + \mathcal{X}^{\|H\|,\|Z\|,\|A\|}_{\alpha,1,\vartheta}(\sigma,0)\|\psi(0)\| + \sum_{m=1}^{d}\|H_m\|\,\mathcal{X}^{\|H\|,\|Z\|,\|A\|}_{\alpha,1,\vartheta}(\sigma,\tau_m)\|\psi(0)\|\right.$$

$$+\sum_{m=1}^{d}\int_{-\tau_m}^{0}\mathcal{X}^{\|H\|,\|Z\|,\|A\|}_{\alpha,\alpha,\vartheta}(\sigma,x+\tau_m)\left\|H_m\left({}_{0^+}^{C}\beth^{\alpha}_{\vartheta}\psi\right)(x) + A_m\psi(x)\right\|d\vartheta(x)$$

$$\left.+\int_0^\sigma\left\|\mathcal{X}^{H,Z,A}_{\alpha,\alpha,\vartheta}(\sigma,x)\right\|\left(\left\|g\big(\mathrm{x},\rho(x)\big)-g(\mathrm{x},0)\right\| + \|g(\mathrm{x},0)\|\right)d\vartheta(x)\right]$$

$$\le R\left[\|\rho_\sigma\| + \mathcal{X}^{\|H\|,\|Z\|,\|A\|}_{\alpha,1,\vartheta}(\sigma,0)\|\psi(0)\| + \sum_{m=1}^{d}\|H_m\|\,\mathcal{X}^{\|H\|,\|Z\|,\|A\|}_{\alpha,1,\vartheta}(\sigma,\tau_m)\|\psi(0)\|\right.$$

$$+\sum_{m=1}^{d}\int_{-\tau_m}^{0}\mathcal{X}^{\|H\|,\|Z\|,\|A\|}_{\alpha,\alpha,\vartheta}(\sigma,x+\tau_m)\left\|H_m\left({}_{0^+}^{C}\beth^{\alpha}_{\vartheta}\psi\right)(x) + A_m\psi(x)\right\|d\vartheta(x)$$

$$+\int_0^\sigma\left\|\mathcal{X}^{H,Z,A}_{\alpha,\alpha,\vartheta}(\sigma,x)\right\|d\vartheta(x)\left\|L_g\right\|_{L^\infty(J,\mathbb{R}^n)}\|\rho\|_C + N_g\int_0^\sigma\left\|\mathcal{X}^{H,Z,A}_{\alpha,\alpha,\vartheta}(\sigma,x)\right\|d\vartheta(x)$$

$$\le R\|\rho_\sigma\| + RR_1 + RR_2\|\rho\|_C.$$

To find such $\varepsilon > 0$ that $\mathcal{K}\rho(\varsigma) \in \mathcal{D}_\varepsilon$, we consider by using $(\boldsymbol{O_1})$ and $(\boldsymbol{O_2})$ and Lemma 2.1,

$$\|\mathcal{K}\rho(\varsigma)\| \le R\left[\mathcal{X}^{\|H\|,\|Z\|,\|A\|}_{\alpha,1,\vartheta}(\sigma,0)\|\psi(0)\| + \sum_{m=1}^{d}\|H_m\|\,\mathcal{X}^{\|H\|,\|Z\|,\|A\|}_{\alpha,1,\vartheta}(\sigma,\tau_m)\|\psi(0)\|\right.$$

$$+\sum_{m=1}^{d}\int_{-\tau_m}^{0}\mathcal{X}^{\|H\|,\|Z\|,\|A\|}_{\alpha,\alpha,\vartheta}(\sigma,x+\tau_m)\left\|H_m\left({}_{0^+}^{C}\beth^{\alpha}_{\vartheta}\psi\right)(x) + A_m\psi(x)\right\|d\vartheta(x)$$

$$+\int_0^\sigma \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,x)\left\|g\big(\mathrm{x},\rho(x)\big)\right\|d\vartheta(x)+\int_0^\sigma \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,x)\left\|v_\rho(x)\right\|d\vartheta(x)\Bigg].$$

When the just-above control estimation is applied, one can acquire

$$\begin{aligned}\|\mathcal{K}\rho(\varsigma)\| &\le \left(1+\vartheta_0^\sigma \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,0)\|S\|R\right)R_1+\left(\vartheta_0^\sigma \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,0)\|S\|R\right)\|\rho_\sigma\|\\ &\quad+\left(1+\vartheta_0^\sigma \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,0)\|S\|R\right)R_2\|\rho\|_C\\ &\le \left(1+\vartheta_0^\sigma \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,0)\|S\|R\right)R_1+\left(\vartheta_0^\sigma \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,0)\|S\|R\right)\|\rho_\sigma\|\\ &\quad+\left(1+\vartheta_0^\sigma \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,0)\|S\|R\right)R_2\varepsilon\coloneqq\varepsilon.\end{aligned}$$

One can easily obtain

$$\varepsilon=\frac{\left(1+\vartheta_0^\sigma \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,0)\|S\|R\right)R_1+\left(\vartheta_0^\sigma \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,0)\|S\|R\right)\|\rho_\sigma\|}{1-\left(1+\vartheta_0^\sigma \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,0)\|S\|R\right)R_2}>0$$

which provides $\mathcal{K}(\mathcal{D}_\varepsilon)\subseteq\mathcal{D}_\varepsilon$. We will separate $\mathcal{K}$ into two distinct operators $\mathcal{K}_1$ and $\mathcal{K}_2$ on $\mathcal{D}_\varepsilon$ like noted below:

$$\begin{aligned}\mathcal{K}_1\rho(\varsigma)&=\left[\mathcal{X}_{\alpha,1,\vartheta}^{H,Z,A}(\varsigma,0)-\sum_{m=1}^{d}H_m\,\mathcal{X}_{\alpha,1,\vartheta}^{H,Z,A}(\varsigma,\tau_m)\right]\psi(0)+\int_0^\varsigma \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma,x)\,Sv_\rho(x)d\vartheta(x)\\ &\quad+\sum_{m=1}^{d}\int_{-\tau_m}^{0}\mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma,x+\tau_m)\left[H_m\left({}_{0^+}^{C}\beth_\vartheta^\alpha\psi\right)(x)+A_m\psi(x)\right]d\vartheta(x)\end{aligned}\tag{14}$$

and

$$\mathcal{K}_2\rho(\varsigma)=\int_0^\varsigma \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma,x)\,g\big(\mathrm{x},\rho(x)\big)d\vartheta(x).\tag{15}$$

<u>*Step 2:*</u> Our first task is to prove $\mathcal{K}_1$ is a contraction. Assume $\rho,u\in\mathcal{D}_\varepsilon$. Keeping $(\boldsymbol{O_1})$ and $(\boldsymbol{O_2})$ in mind, we get

$$\begin{aligned}\left\|v_\rho(\varsigma)-v_u(\varsigma)\right\|&\le\int_0^\sigma\left\|\mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x)\right\|\left\|g\big(\mathrm{x},\rho(x)\big)-g(\mathrm{x},0)\right\|d\vartheta(x)\\ &\le\int_0^\sigma\left\|\mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x)\right\|d\vartheta(x)\left\|L_g\right\|_{L^\infty(J,\mathbb{R}^n)}\|\rho-u\|_C\\ &\le RR_2\|\rho-u\|_C.\end{aligned}$$

So

$$\|\mathcal{K}_1\rho(\varsigma) - \mathcal{K}_1 u(\varsigma)\| \leq \int_0^{\sigma} \left\|\mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\sigma,x)\right\| \|S\| \left\|v_\rho(x) - v_u(x)\right\| d\vartheta(x)$$

$$\leq \vartheta_0^{\sigma} \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,0)\|S\|RR_2\|\rho - u\|_C.$$

Since (11), $\vartheta_0^{\sigma} \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,0)\|S\|RR_2 < 1$, it shows a contradiction of $\mathcal{K}_1$.

<u>*Step 3:*</u> The rest thing to do is that $\mathcal{K}_2$ is a both continuous and compact function. Assume $\rho_n \in \mathcal{D}_\varepsilon$ with $\rho_n \to \rho$ in $\mathcal{D}_\varepsilon$. $(\boldsymbol{O_2})$ ensures $g(\varsigma, \rho_n(\varsigma)) \to g(\varsigma, \rho(\varsigma))$ in C. In the light of the dominated convergence theorem

$$\|\mathcal{K}_2\rho(\varsigma) - \mathcal{K}_2\rho_n(\varsigma)\| \leq \int_0^{\varsigma} \left\|\mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma,x)\right\| \left\|g\big(x,\rho(x)\big) - g\big(x,\rho_n(x)\big)\right\| d\vartheta(x)$$

goes to zero as $n \to \infty$. Thus $\mathcal{K}_2$ is a continuous function in $\mathcal{D}_\varepsilon$. We have to prove that $\mathcal{K}_2(\mathcal{D}_\varepsilon) \subseteq C$ is an equicontinuous and uniformly bounded operator so as to verify $\mathcal{K}_2$ is compact. For an arbitrary $\rho \in \mathcal{D}_\varepsilon$, $0 < \varsigma < \varsigma + h < \sigma$,

$$\mathcal{K}_2\rho(\varsigma + h) - \mathcal{K}_2\rho(\varsigma) = \int_{\varsigma}^{\varsigma+h} \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma + h, x)\, g\big(\mathrm{x}, \rho(x)\big) d\vartheta(x)$$

$$+ \int_0^{\varsigma} \left(\mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma + h, x) - \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma, x)\right) g\big(\mathrm{x}, \rho(x)\big) d\vartheta(x)$$

Set the following notations:

$$\Delta_1 = \int_{\varsigma}^{\varsigma+h} \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma + h, x)\, g\big(\mathrm{x}, \rho(x)\big) d\vartheta(x),$$

$$\Delta_2 = \int_0^{\varsigma} \left(\mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma + h, x) - \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma, x)\right) g\big(\mathrm{x}, \rho(x)\big) d\vartheta(x).$$

Since

$$\|\mathcal{K}_2\rho(\varsigma + h) - \mathcal{K}_2\rho(\varsigma)\| \leq \|\Delta_1\| + \|\Delta_2\|,$$

it is enough to demonstrate that $\|\Delta_i\| \to 0$, $h \to 0$, $i = 1,2$. By means of a simple calculation

$$\|\Delta_1\| \le \int_{\varsigma}^{\varsigma+h} \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma+h,x)\, d\vartheta(x) \|L_g\|_{L^\infty(J,\mathbb{R}^n)} \|\rho\|_C$$

$$+N_g \int_{\varsigma}^{\varsigma+h} \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma+h,x)\, d\vartheta(x)$$

$$\le \vartheta_{\varsigma}^{\varsigma+h} \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(h,0) \|L_g\|_{L^\infty(J,\mathbb{R}^n)} \|\rho\|_C + N_g \vartheta_{\varsigma}^{\varsigma+h} \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(h,0)$$

and

$$\|\Delta_2\| \le \int_0^{\varsigma} \left\|\mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma+h,x) - \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma,x)\right\| d\vartheta(x) \|L_g\|_{L^\infty(J,\mathbb{R}^n)} \|\rho\|_C$$

$$+N_g \int_0^{\varsigma} \left\|\mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma+h,x) - \mathcal{X}_{\alpha,\alpha,\vartheta}^{H,Z,A}(\varsigma,x)\right\| d\vartheta(x).$$

Thus, $\|\Delta_i\| \to 0$, $h \to 0$, $i = 1,2$. Consequently, we acquire for $\rho \in \mathcal{D}_\varepsilon$,

$$\|\mathcal{K}_2\rho(\varsigma+h) - \mathcal{K}_2\rho(\varsigma)\| \to 0, \qquad h \to 0$$

$\mathcal{K}_2(\mathcal{D}_\varepsilon)$ is bounded due to the following inequality as an upper bound

$$\|\mathcal{K}_2\rho(\varsigma)\| \le \vartheta_0^{\sigma} \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,0) \|L_g\|_{L^\infty(J,\mathbb{R}^n)} \|\rho\|_C + N_g \vartheta_0^{\sigma} \mathcal{X}_{\alpha,\alpha,\vartheta}^{\|H\|,\|Z\|,\|A\|}(\sigma,0).$$

So Arzela-Ascoli theorem provides $\mathcal{K}_2(\mathcal{D}_\varepsilon)$ is a continuous operator which is compact in C. This completes the proof.

Remark 2.1 All theoretical findings for selecting $d = 1$ and $\tau_1 = \tau$ reduce to those ones of (Muthuvel et al., 2023).

3. NUMERICAL EXAMPLES

Now we illustrate theoretical outcomes.

Example 3.1 We consider the below equation to exemplify the controllability of the homogeneous case:

$$ {}_{0^+}^{C}\beth_{\vartheta}^{\alpha}[\rho(\varsigma) - H_1\rho(\varsigma-\tau_1)] = Z\rho(\varsigma) + \sum_{k=1}^{2} A_k\rho(\varsigma-\tau_k) + Sv(\varsigma), \varsigma \in [0,2] \tag{16}$$

$$\rho(\varsigma) = \psi(\varsigma), - \ -0.3 \le \varsigma \le 0,$$

where

$$H_1 = \begin{pmatrix} 0.8 & 0.1 & 0.2 \\ 0.3 & 0.4 & 0.6 \\ 0 & 0.7 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 0.1 & 0 & 0.2 \\ 0 & 0.3 & 0.3 \\ 0.25 & 0 & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0.31 & 0.2 & 0 \\ 0.45 & 0.74 & 0.35 \\ 0.1 & 0.8 & 0.02 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0.72 & 0.36 \\ 0.83 & 0.64 & 0.78 \\ 0.21 & 0.08 & 0.12 \end{pmatrix},$$

$$S = \begin{pmatrix} 0.51 & 0 & 0 \\ 0.423 & 0.14 & 0.86 \\ 0.8 & 0.6 & 0.2 \end{pmatrix},$$

$\vartheta(\varsigma) = 2\varsigma,\ \alpha = 0.7,\ r_1 = 0.1,\ r_2 = 0.3$ and $\psi(\varsigma) = [\varsigma + 3\ 4\varsigma + \pi\ 5]^T \in \mathbb{R}^3$. The multi-delayed neutral Gramian matrix is

$$\mathfrak{M}_{0.7,0.3,2t}[0,1] = \int_0^1 \chi_{2t,0.7,0.7}^{A_1,B,F_1+F_2}(1,s) S S^T \chi_{2t,0.7,0.7}^{A_1,B,F_1+F_2}(1,s) d\vartheta(s)$$
$$= \begin{pmatrix} 32.507 & 60.222 & 38.525 \\ 60.222 & 111.762 & 71.451 \\ 38.525 & 71.451 & 5.702 \end{pmatrix}$$

We eaisly compute its determinant $\left|\mathfrak{M}_{0.7,0.3,2t}[0,1]\right| = 0.076$. Thus, it is nonsingular. Equation (16) is relatively controllable based on Theorem 3,

Example 2 We consider the below equation so as to illustrate the semilinear fractional differential neutral multi-delayed equation for $\varsigma \in [0,6]$

$$\begin{gathered} {}_{0^+}^{C}\beth_{\vartheta}^{\alpha}[\rho(\varsigma) - H_1\rho(\varsigma - 0.4)] = Z\rho(t) + A_1\rho(\varsigma - 0.5) + Sv(\varsigma) + g\big(\varsigma, \rho(\varsigma)\big), \\ \rho(\varsigma) = \psi(\varsigma), \qquad -0.5 \le \varsigma \le 0, \end{gathered} \tag{17}$$

where

$$H_1 = \begin{pmatrix} 0.7 & 0 \\ 0.3 & 0.2 \end{pmatrix}, \qquad Z = \begin{pmatrix} 0 & 0.9 \\ 0.2 & 0.8 \end{pmatrix},$$
$$A_1 = \begin{pmatrix} 0.35 & 0.65 \\ 0 & 0.7 \end{pmatrix}, \quad S = \begin{pmatrix} 0.8 & 0 \\ 0.4 & 0.6 \end{pmatrix},$$

and $\psi(\varsigma) = [1\ 5]^T$ and $g\big(\varsigma, \rho(\varsigma)\big) = \left[\frac{\tan^{-1}\rho(\varsigma)}{(\pi^2\varsigma)^2}\ \frac{\sin\rho(\varsigma)}{\pi^6\varsigma}\right]^T$. Let's investigate the assumptions for system (17) one by one,

$$\mathfrak{M}_{\tau,\alpha,\vartheta}[0,\varsigma] = \mathfrak{M}_{0.5,0.5,t+1}[0,3] = \begin{pmatrix} 68.896 & 63.956 \\ 63.956 & 59.944 \end{pmatrix}$$

and

$$= \sqrt{\left\|\mathfrak{M}_{0.5,0.5,t+1}[0,3]\right\|R} = \left\|\mathfrak{M}_{\vartheta,C}\right\|^{-1}_{B\left(\mathbb{R}^n,L^2,(J,\mathbb{R}^n)/ker\mathfrak{M}_{\vartheta,c}\right)} = 6.2874$$

which guarantees that the inverse operator $\mathfrak{M}^{-1}_{\vartheta,C}$ exist, so the operator $\mathfrak{M}_{\vartheta,C}$ holds (O_1). $g : [0,3] \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is continuous. For any $\rho, \omega \in \mathbb{R}^n$

$$\left\|\left[\frac{\tan^{-1}\rho(\varsigma)}{(\pi^2\varsigma)^2}\ \frac{\sin\rho(\varsigma)}{\pi^6\varsigma}\right]^T - \frac{\tan^{-1}\omega(\varsigma)}{10(\pi^2\varsigma)^2}\ \frac{\sin\omega(\varsigma)}{20(\pi\varsigma)^6}\right\| \leq L_g(\varsigma)\|\rho(\varsigma) - \omega(\varsigma)\|, \varsigma \in [0,3]$$

where $L_g(\varsigma) = \frac{1}{10\pi^4\varsigma} \in L^\infty(J, \mathbb{R}^+)$. So (O_2) is satisfied for equation (17).

$$R_2\left(1 + \vartheta_0^3 \chi_{0.5,0.5,t+1}^{\|A\|,\|B\|,\|F\|}(3,0)\|S\|R\right) = 0.6320 < 1$$

Thus, the inequality (11) also is satisfied. As a result, Theroem 4 gives that equation (17) is relatively controllable via the control $\upsilon_\rho(\varsigma)$ which can be obtained from (12).

Remark 2 We would like to state that under the permutable matrices by choosing $d = \alpha = \beta = 1$ and $\vartheta(\varsigma) = \varsigma$, the obtained results overlaps with those of (You et al., 2021).

4. CONCLUSION

We identify a notion of the ϑ-Gramian matrix so as to demonstrate that the linear multi-delayed neutral equation consisting of the Caputo derivative w.r.t another function is relatively controllable, and offer the both necessary and sufficient circumstances for the linear system. We acquire the controllability outcomes for the semi-linear multi-delayed neutral equation having commutative coefficients consisting of the Caputo derivative w.r.t another function by means of the fixed point theorem of the Krasnoselskii.

Moreover, this paper in terms of relative controllability of fractional or ordinary differential system and the offered findings contains many distinct sorts of extensive papers(Pospısil, 2017; Pospisil & Skripkova, 2015; Sontag, 2013; You et al., 2020), etc and more undone studies which include all obtained results above since for some special choices of ϑ, one can acquire the traditional Caputo FD(Samko et al., 1993), the well-known Hadamard FD(Kilbas et al., 2006) the Caputo–Erdélyi–Kober FD given in (Luchko & Trujillo, 2007) and the Caputo–Hadamard FD given in (Gambo et al., 2014; Jarad et al., 2012).

The next possible study could be devoted to working Lyapunov, finite-time, exponential stabilities of the ϑ-Caputo fractional differential multi-delayed neutral system with non-permutable coefficients. Another direction for additional studies is to investigate approximate controllability, Ulam-Hyers stability, and asymptotic stability results for ϑ-fractional functional evolution equations.

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CHAPTER 3

ON THE PARAMETERIZED DIFFERENTIAL TRANSFORMATION METHOD

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INTRODUCTION

Various phenomena in natural science are usually modeled by the BVPs for ODEs and PDEs. In most cases finding an analytical solutions to such problems is very difficult. Alternatively, numerical or semi-analytical methods may provide approximate solutions rather than analytical ones. The DTM is semi- analytical method is semi-analytical method for of linear and nonlinear BVPs for various type of differential equations. As opposed to the Taylor expansion the DT method provides approximate or exact solutions to BVPs without the need to calculate higher derivatives of data functions.

The classical DTM was first developed by Zhou in the study of problems appearing in electrical circuit analysis (Zhou, 1986). This method was later developed in different directions by many scientists (see, (Chiou and Tzeng,1996), (Chen and Ho, 1999), (Ayaz, 2004)). Ertürk and Momani used the DTM and ADM to get an approximate solutions for fourth -order BVPs. This work also provides a numerical comparison of DTM and ADM-solutions (Ertürk and Momani, 2007). Wazwaz has developed a new generalization of decomposition method for obtaining an approximate solution to a special type high-order BVP's (Wazwaz, 2002). In some works, the classical DTM was modified so that it can be applied to study not only single-interval BVPs, but also many-interval boundary-value-transmission problems (see, for example (Arslan, 2022), (Mukhtarov, Yücel and Aydemir, 2021), (Mukhtarov, Çavuşoğlu and Olğar, 2019)). In this work we shall propose a new technique, the so-called parameterized DTM (P DTM), which is an modification and generalization of classic DTM, since for the concrete values of the parameter the PDTM reduces to the classic DTM.

1. THE PARAMETERIZED DTM

In below we have defined a new modification of the classical differential transformation technique which you can find approximate, and in some cases even exact solutions not only regular initial and/or boundary value problems, as well as similar problems involving an internal singularity.

Suppose that $f: R \to R$, is an analytic function, that is $f(x)$ can be expanded in a Taylor series. Denote

$$Y_s(f, x_0) := \frac{1}{s!} f^{(s)}(x_0), \qquad s = 0,1,2 \ldots$$

Definition 1. Let $\alpha \in [0,1]$ be any real number. The sequence

$$D_\alpha(f) := (D(f, \alpha; n))$$

is called α −parameterized DTM the original function $f(x)$ at the two points $x = a$ and $x = b$, where
$-\infty < a < b < \infty$, and $D(f, \alpha; s)$ is defined by

$$D(f, \alpha; s) \coloneqq \alpha Y_s(f, a) + (1 - \alpha) Y_s(f, b).$$

Definition 2. Given $\tilde{C} = (c_0, c_1, c_2, \dots)$. Suppose the series

$$\mathrm{Z}_\alpha(C, \mathrm{x}) := \sum_{s=1}^{\infty} C_s(\mathrm{x} - (\alpha\mathrm{a} + (1 - \alpha)\mathrm{b})^k$$

is convergens in R. Then the function $\mathrm{Z}_\alpha(D_\alpha(f), \mathrm{x})$ is called parametrized DTM.

Remark 1. The equality $Z_\alpha(D_\alpha(f), x) = f(x)$ is not satisfied for all analytical functions. However, the following Theorem is true.

Theorem 1. If $\alpha = 0$ or $\alpha = 1$, then $Z_\alpha = D_\alpha^{-1}$, that is the equalities $Z_0(D_0(f), x) = f(x)$ and $Z_1(D_1(f), x) = f(x)$ are satisfied for all analytic functions $f(x)$.

Corollary 1. Let $\alpha = 0$ or $\alpha = 1$. Then the proposed parameterized DTM reduces to the well-know differential transformation at the points $x = b$ and $x = a$ respectively. Likewise, in these cases $\alpha -$ P DTM reduce to the classical inverse DTM.

Definition 3. The finite sum

$$s_{\alpha,n}(t) \coloneqq \sum_{n=0}^{N} (s_\alpha(c, d))_n(s)\big(t - (\alpha c + (1 - \alpha)d)\big)^n,$$

That is the N-th partial sum of the inverse differential transformation is said to be an N-th parameterized approximation of the original function.

Theorem 2. If $f(x) = const$, then the equalities

$$Z_\alpha(D_\alpha(f), x) = f(x) \;\; and \; Z_\alpha\big(D_{\alpha,n}(f), x\big) = f(x), \;\; n = 0,1, \dots$$

are hold.

Theorem 3. Let $\beta \in R$ be any real number. Then

a) $D_\alpha(\beta f) = \beta D_\alpha(f)$
b) $Z_\alpha(D_\alpha(\beta f), x) = \beta Z_\alpha(D_\alpha(f), x)$
c) $Z_\alpha\big(D_{\alpha,n}(\beta f), x\big) = \beta Z_\alpha\big(D_{\alpha,n}(f), x\big), \;\; n = 0,1,2, \dots$

Theorem 4.

a) $D_\alpha(f \pm g) = D_\alpha(f) \pm D_\alpha(g)$

b) $Z_\alpha(D_\alpha(f \pm g), x) = Z_\alpha(D_\alpha(f), x) \pm Z_\alpha(D_\alpha(g), x)$

c) $Z_\alpha\big(D_{\alpha,n}(f \pm g), x\big) = Z_\alpha\big(D_{\alpha,n}(f), x\big) \pm Z_\alpha\big(D_{\alpha,n}(g), x\big),$
$$for\ all\ n = 0,1,2, \ldots$$

Theorem 5. Let $g(x) = \frac{d^m f(x)}{dx^m}$, $n \in N$. Then

a) $D(g, \alpha; s) = \frac{(s+m)!}{s!} D(f, \alpha; s+m)$

b) $\frac{d^m}{dx^m} \tilde{f}_{\alpha,n}(x) = \sum_{s=0}^{n} \frac{(s+m)!}{k!} D(f, \alpha; s+m)(x - x_\alpha)^s,$
$where\ x_\alpha = \alpha a + (1-\alpha)b.$

Theorem 6. If $f(x) = g(x)h(x)\ then$
$D(f, \alpha; s) = \sum_{m=0}^{s} [\alpha Y_m(g; a) Y_{s-m}(h; a) + (1-\alpha) Y_m(g; b) Y_{s-m}(h; b)].$

Theorem 7. Let $f(x) = x^m$, $m \in N$. Then

$$D(f, \alpha; s) = \begin{cases} \binom{m}{s}(\alpha a^{m-s} + (1-\alpha) b^{m-s}) & for\ s < m \\ 1 & for\ s = m \\ 0 & for\ s > m \end{cases}$$

2. NUMERICAL RESULTS

2.1 Example

Consider the following HOBVP

$$y^{(4)}(t) = 4e^t + y(t), 0 < t < 1 \qquad (2.1.1)$$

subject to the BCs

$$y|_0 = 1, \qquad \left.\frac{d^2 y}{dt^2}\right|_0 = 3, \qquad y|_1 = 2e, \qquad \left.\frac{d^2 y}{dt^2}\right|_1 = 4e \qquad (2.1.2)$$

This problem has an exact solution, given by $y(t) = (1+t)e^t$.

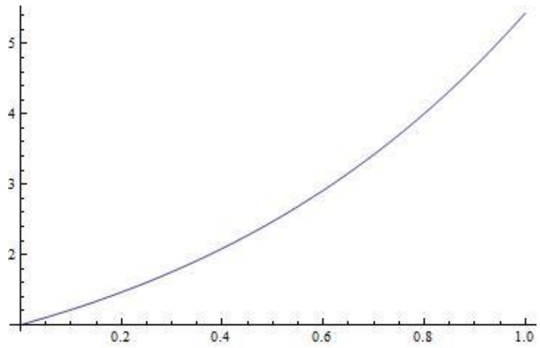

Figure 1: The graph of the function $y(t) = (1+t)e^t$.

If it is applied *PDT* to both sides of (2.1.1), then we obtain

$$(n+1)(n+2)(n+3)(n+4)\big(S_\alpha(0,1)\big)_{n+4}(y) = \left(\big(S_\alpha(0,1)\big)_n(y) + \frac{4}{n!}\right) \tag{2.1.3}$$

Therefore, from the definition of PDT,

$$\mathrm{y}_\alpha(\mathrm{t}) = \sum_{\mathrm{n}=0}^{\infty} \big(\mathrm{S}_\alpha(0,1)\big)_{\mathrm{n}}(\mathrm{y})(\mathrm{t}-\mathrm{t}_\alpha)^{\mathrm{n}}$$

and

$$y''_\alpha(t) = \sum_{n=0}^{\infty} \big(S_\alpha(0,1)\big)_n(y)n(n-1)(t-t_\alpha)^{n-2}$$

Moreover, for the BCs $y|_0 = 1,\ \frac{d^2y}{dt^2}\Big|_0 = 3,\ y|_1 = 2e,\ \frac{d^2y}{dt^2}\Big|_1 = 4e$

$$y_\alpha(0) = \sum_{n=0}^{N} \big(S_\alpha(0,1)\big)_n(y)(\alpha-1)^n = 1$$

$$y''_\alpha(0) = \sum_{n=0}^{N} \big(S_\alpha(0,1)\big)_n(y)n(n-1)(\alpha-1)^{n-2} = 3$$

$$y_\alpha(1) = \sum_{n=0}^{N} \big(S_\alpha(0,1)\big)_n(y)(\alpha)^n = 2e$$

$$y''_\alpha(1) = \sum_{n=0}^{N} \big(S_\alpha(0,1)\big)_n(y)n(n-1)(\alpha)^{n-2} = 4e$$

respectively. Here, let $\left(S_\alpha(0,1)\right)_0(y) = \rho$, $\left(S_\alpha(0,1)\right)_1(y) = \sigma$, $\left(S_\alpha(0,1)\right)_2(y) = \tau$, and $\left(S_\alpha(0,1)\right)_3(y) = \omega$, then substituting in the recursive relation (5), we can calculate the other terms of the PDT as

$$\left(S_\alpha(0,1)\right)_4(y) = \frac{1}{3!} + \frac{\rho}{4!}, \quad \left(S_\alpha(0,1)\right)_5(y) = \frac{4}{5!} + \frac{\sigma}{5!},$$
$$\left(S_\alpha(0,1)\right)_6(y) = \frac{4}{6!} + \frac{\tau}{6!}, \quad \left(S_\alpha(0,1)\right)_7(y) = \frac{4}{7!} + \frac{6\omega}{7!},$$
$$\left(S_\alpha(0,1)\right)_8(y) = \frac{8}{8!} + \frac{\rho}{8!}, \quad \left(S_\alpha(0,1)\right)_9(y) = \frac{8}{9!} + \frac{\sigma}{9!},$$
$$\left(S_\alpha(0,1)\right)_{10}(y) = \frac{8}{10!} + \frac{2\tau}{10!}, \quad \left(S_\alpha(0,1)\right)_{11}(y) = \frac{8}{11!} + \frac{6\omega}{11!},$$
$$\left(S_\alpha(0,1)\right)_{12}(y) = \frac{12}{12!} + \frac{\rho}{12!}, \quad \left(S_\alpha(0,1)\right)_{13}(y) = \frac{12}{13!} + \frac{\sigma}{13!},$$
$$\left(S_\alpha(0,1)\right)_{14}(y) = \frac{12}{14!} + \frac{2\tau}{14!}, \quad \left(S_\alpha(0,1)\right)_{15}(y) = \frac{12}{15!} + \frac{6\omega}{15!}, \ldots$$

Hence, the parameterized series solution $y_\alpha(t)$ is evaluated up to $N = 15$:

$$
\begin{aligned}
y_\alpha(t) &= \sum_{n=0}^{15} \left(S_\alpha(0,1)\right)_n (y)(t - t_\alpha)^n \\
&= \rho + \sigma(t-1+\alpha) + \tau(t-1+\alpha)^2 + \omega(t-1+\alpha)^3 \\
&+ \left(\frac{1}{3!} + \frac{\rho}{4!}\right)(t-1+\alpha)^4 + \left(\frac{4}{5!} + \frac{\sigma}{5!}\right)(t-1+\alpha)^5 \\
&+ \left(\frac{4}{6!} + \frac{\tau}{6!}\right)(t-1+\alpha)^6 + \left(\frac{4}{7!} + \frac{6\omega}{7!}\right)(t-1+\alpha)^7 \\
&+ \left(\frac{8}{8!} + \frac{\rho}{8!}\right)(t-1+\alpha)^8 + \left(\frac{8}{9!} + \frac{\sigma}{9!}\right)(t-1+\alpha)^9 \\
&+ \left(\frac{8}{10!} + \frac{2\tau}{10!}\right)(t-1+\alpha)^{10} + \left(\frac{8}{11!} + \frac{6\omega}{11!}\right)(t-1+\alpha)^{11} \\
&+ \left(\frac{12}{12!} + \frac{\rho}{12!}\right)(t-1+\alpha)^{12} + \left(\frac{12}{13!} + \frac{\sigma}{13!}\right)(t-1+\alpha)^{13} \\
&+ \left(\frac{12}{14!} + \frac{2\tau}{14!}\right)(t-1+\alpha)^{14} + \left(\frac{12}{15!} + \frac{6\omega}{15!}\right)(t-1+\alpha)^{15}
\end{aligned}
$$

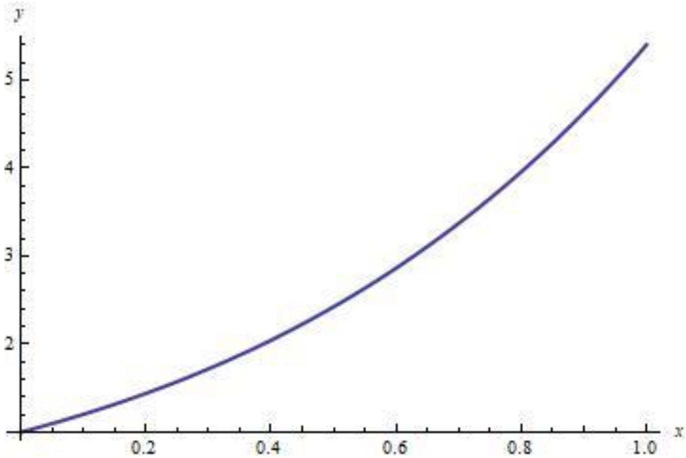

Figure 2. The PDTM-solution of (2.1.1)-(2.1.2) for $\alpha = 0,5$.

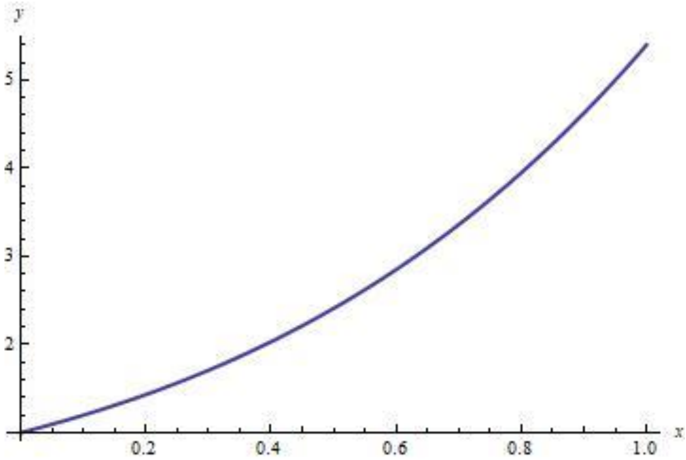

Figure 3. The PDTM-solution of (2.1.1)-(2.1.2) for $\alpha = 0{,}2$

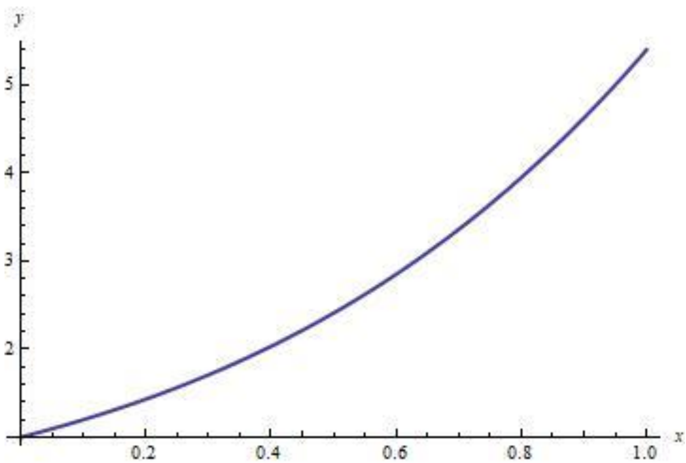

Figure 4. The PDTM-solution of (2.1.1)-(2.1.2) for $\alpha = 0{,}999$.

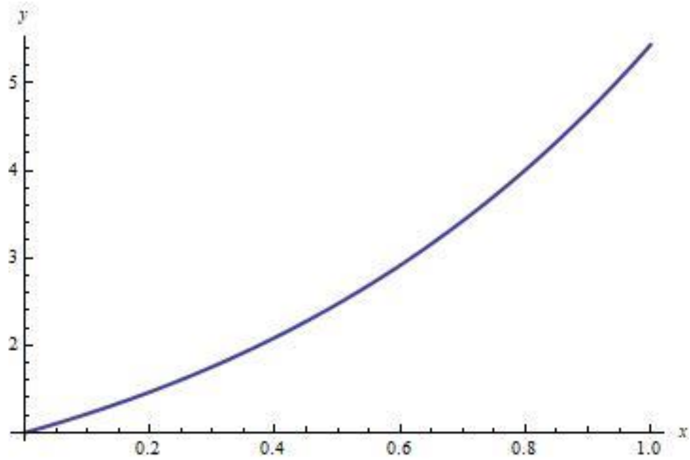

Figure 5. The ADM solution of (2.1.1)-(2.1.2)

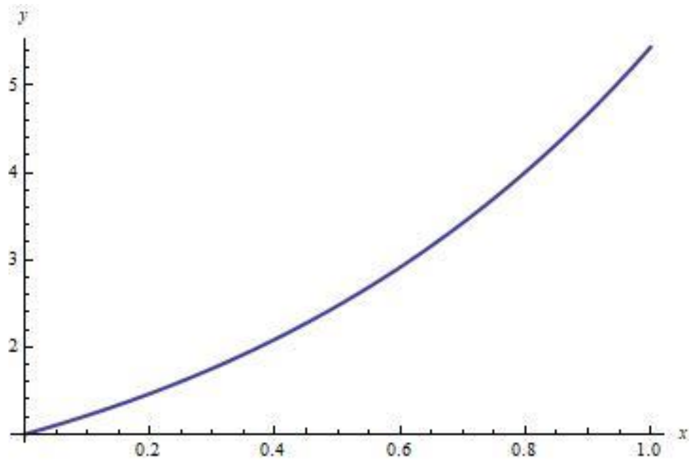

Figure 6. The DTM solution of (2.1.1)-(2.1.2)

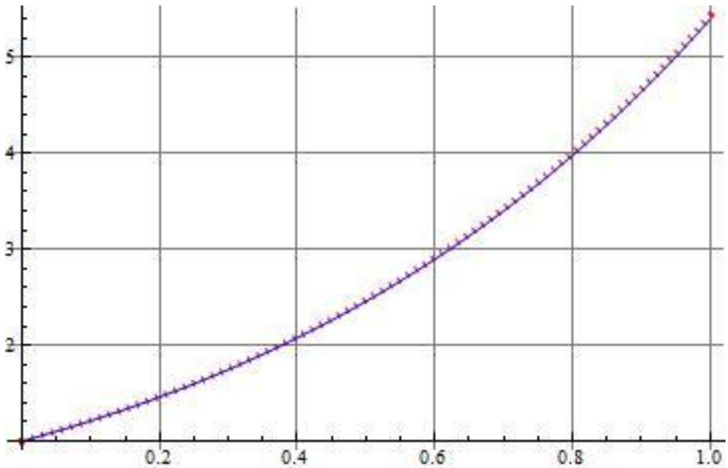

Figure 7. The analytic solution (red line) is compared with the DTM-solution (blue line) for the problem (2.1.1)-(2.1.2)

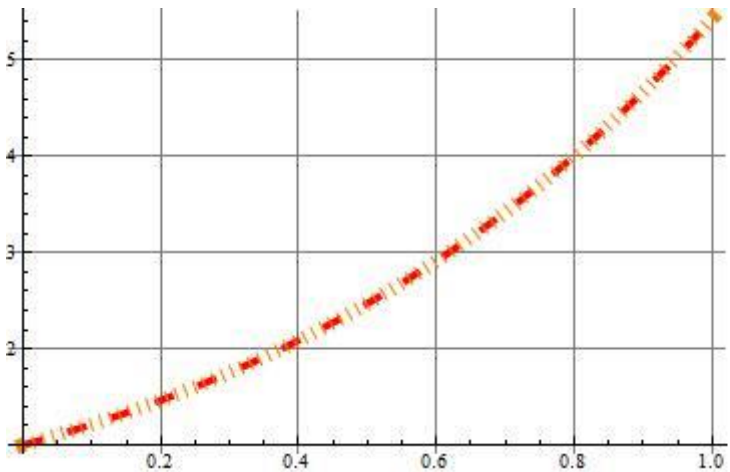

Figure 8. The analytic solution (red line) is compared with the PDTM-solution (orange line)

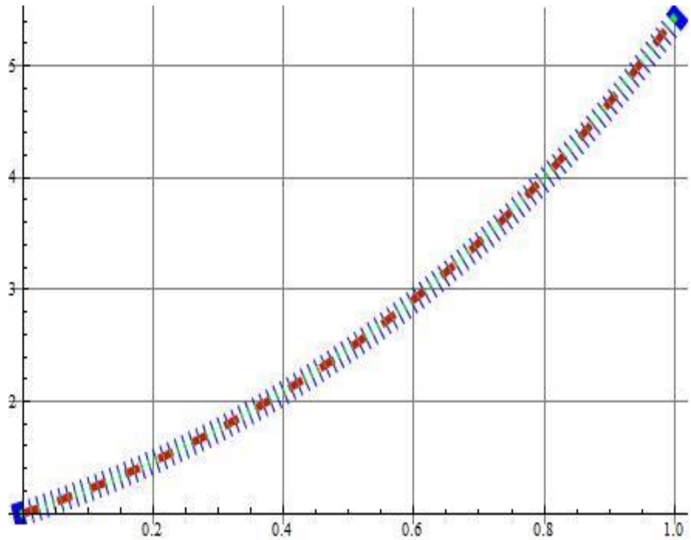

Figure 9. Graphs of the analytical solution (red line), ADM-solution (green line) and PDTM-solution (blue line).

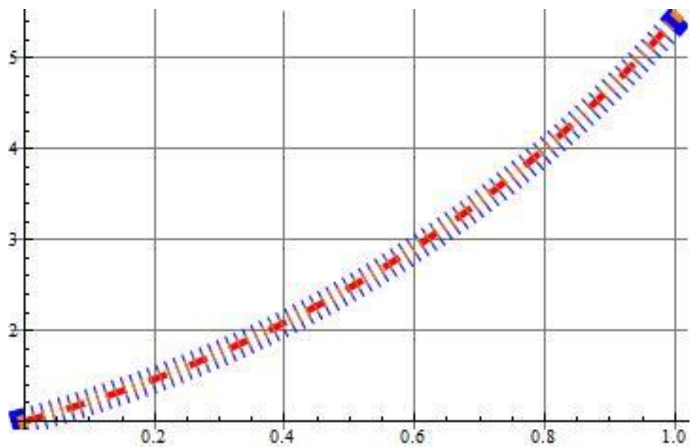

Figure 10. Graphs of the analytical solution (red line) DTM-solution (orange line) and PDTM-solution (blue line)

2.2 Example

Given

$$y''(x) + 2y'(x) + y(x) = e^{-x}, \qquad x \in [-1,0] \tag{2.2.1}$$

with the nonhomogeneous boundary condition

$$y|_{-1} = 1 \text{ and } y|_0 = 0 \tag{2.2.2}$$

We can write that the analytic solution of this problem is

$$y(x) = \frac{1}{2} e^{-1-x} x(-2 + e + ex) \tag{2.2.3}$$

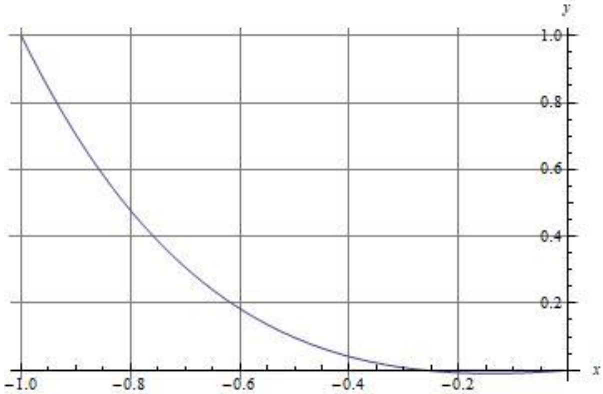

Figure 11. Graph of the exact solution of the problem (2.2.1)-(2.2.3)

By applying PDTM we have

$$(s+1)(s+2)D(y,\alpha;s+2) = -2(s+1)D(y,\alpha;s+1) - D(y,\alpha;s) + \frac{(-1)^s}{s!} \tag{2.2.4}$$

Consequently,

$$y_\alpha(x) = \sum_{s=0}^{\infty} D(y,\alpha;s)(x - x_\alpha)^s$$

Using $y|_{-1} = 1$ and $y|_0 = 0$, we get

$$y_\alpha(-1) = \sum_{s=0}^{P} D(y,\alpha;s)(\alpha - 1)^s = 1 \tag{2.2.5}$$

and

$$y_\alpha(0) = \sum_{s=0}^{P} D(y,\alpha;s)(\alpha)^s = 0 \tag{2.2.6}$$

Let $D_0 = D(y,\alpha;0)$ and $D_1 = D(y,\alpha;1)$. Then we have

$$\mathrm{D}(\mathrm{y},\alpha;3)=\frac{1}{3!}[3D_1+2D_0-3],$$
$$\mathrm{D}(\mathrm{y},\alpha;4)=\frac{1}{12}\left[-2D_1-\frac{3}{2}D_0+3\right],$$
$$\mathrm{D}(\mathrm{y},\alpha;5)=\frac{1}{20}\left[\frac{5}{6}D_1+\frac{2}{3}D_0-\frac{10}{6}\right],$$
$$\mathrm{D}(\mathrm{y},\alpha;6)=\frac{1}{30}\left[-\frac{3}{12}D_1-\frac{5}{24}D_0+\frac{15}{24}\right],$$
$$\vdots$$

Taking $P=6$, yields

$$\begin{aligned} y(x,\alpha) &= \sum_{k=0}^{6} D(y,\alpha;k)(x-x_\alpha)^k \\ &= D_0+(x+\alpha)D_1+(x+\alpha)^2\frac{1}{2}[-2D_1-D_0+1] \\ &\quad +(x+\alpha)^3\frac{1}{3!}[3D_1+2D_0-3] \\ &\quad +(x+\alpha)^4\frac{1}{12}\left[-2D_1-\frac{3}{2}D_0+3\right] \\ &\quad +(x+\alpha)^5\frac{1}{20}\left[\frac{5}{6}D_1+\frac{2}{3}D_0-\frac{10}{6}\right] \\ &\quad +(x+\alpha)^6\frac{1}{30}\left[-\frac{3}{12}D_1-\frac{5}{24}D_0+\frac{15}{24}\right] \end{aligned} \tag{2.2.7}$$

where $x_\alpha=-\alpha$ and $D(y,\alpha;0)=D_0, D(y,\alpha;1)=D_1$. Thus, from (2.2.7),

$$\begin{aligned} y(-1,\alpha) &= D_0+D_1(-1+\alpha)+\frac{1}{2}(1-D_0-2D_1)(-1+\alpha)^2+\frac{1}{6}(-3 \\ &\quad +2D_0+3D_1)(-1+\alpha)^3+\frac{1}{12}(3-\frac{3D_0}{2}-2D_1)(-1+\alpha)^4 \\ &\quad +\frac{1}{20}(-\frac{5}{3}+\frac{2D_0}{3}+\frac{5D_1}{6})(-1+\alpha)^5+\frac{1}{30}(\frac{5}{8}+\frac{5D_0}{24} \\ &\quad -\frac{D_1}{4})(-1+\alpha)^6=1 \end{aligned}$$

and

$$\begin{aligned} y(0,\alpha) &= D_0+D_1\alpha+\frac{1}{2}(1-D_0-2D_1)\alpha^2+\frac{1}{6}(-3+2D_0+3D_1)\alpha^3 \\ &\quad +\frac{1}{12}(3-\frac{3D_0}{2}-2D_1)\alpha^4+\frac{1}{20}(-\frac{5}{3}+\frac{2D_0}{3}+\frac{5D_1}{6})\alpha^5 \\ &\quad +\frac{1}{30}(\frac{5}{8}+\frac{5D_0}{24}-\frac{D_1}{4})\alpha^6=0 \end{aligned}$$

Using (2.2.2) we get

$$D_0 = -(15(-2040\alpha + 17496\alpha^2 - 27612\alpha^3 + 23404\alpha^4 - 13029\alpha^5 + 4977\alpha^6 - 1274\alpha^7 + 210\alpha^8 - 21\alpha^9 + \alpha^{10}))/(-234720 + 467400\alpha - 457320\alpha^2 + 278940\alpha^3 - 99200\alpha^4 - 649\alpha^5 + 23905\alpha^6 - 13250\alpha^7 + 3790\alpha^8 - 625\alpha^9 + 49\alpha^{10}$$

and

$$D_1 = (5(-6120 + 69840\alpha - 77088\alpha^2 + 33108\alpha^3 + 3231\alpha^4 - 14236\alpha^5 + 11276\alpha^6 - 5068\alpha^7 + 1471\alpha^8 - 260\alpha^9 + 22\alpha^{10}))/(-234720 + 467400\alpha - 457320\alpha^2 + 278940\alpha^3 - 99200\alpha^4 - 649\alpha^5 + 23905\alpha^6 - 13250\alpha^7 + 3790\alpha^8 - 625\alpha^9 + 49\alpha^{10})$$

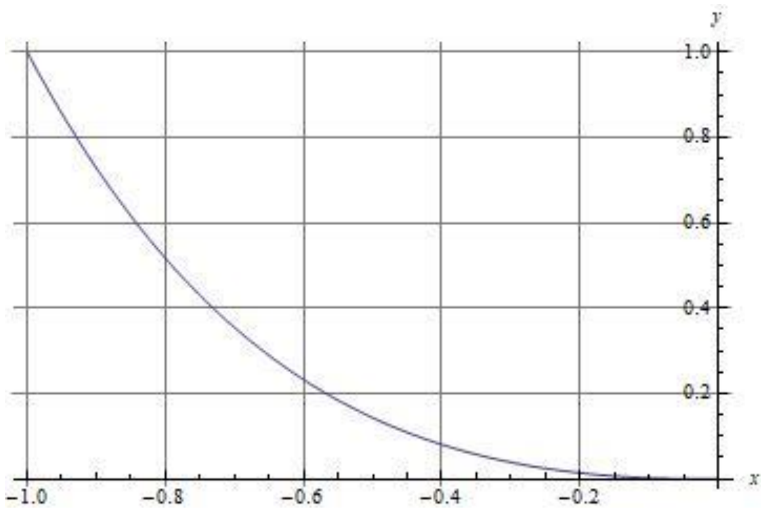

Figure 12. The PDTM- solution for $\alpha = 0{,}25$.

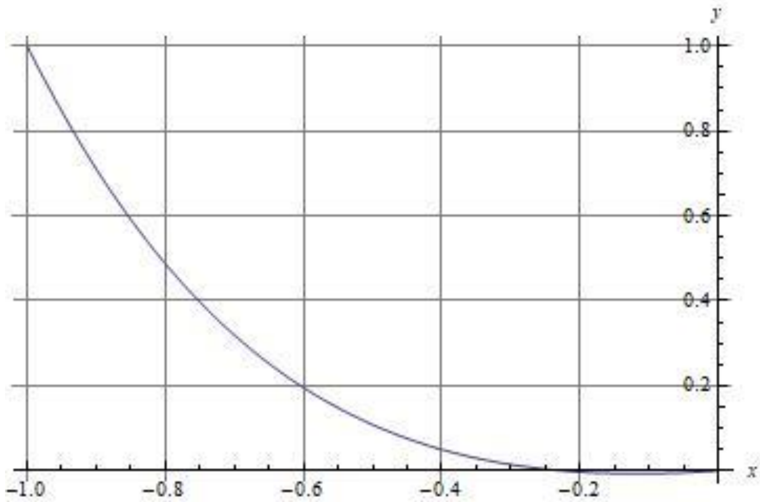

Figure 13. The PDTM- solution for $\alpha = 0{,}05$.

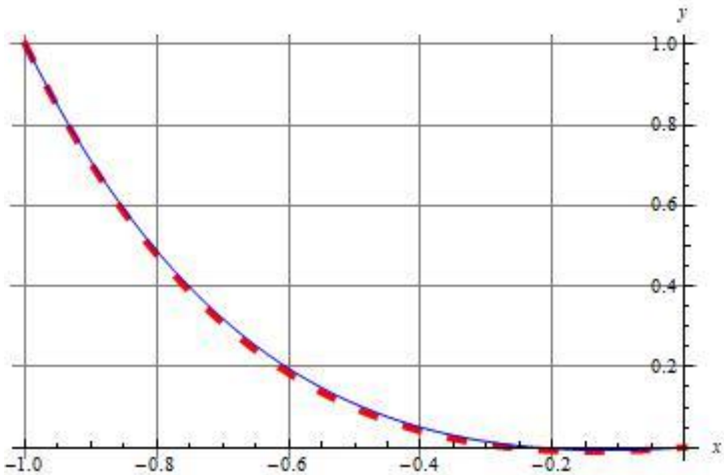

Figure 14. The analytic solution (red line) and the PDTM-solution (blue line) for $\alpha = 0{,}05$.

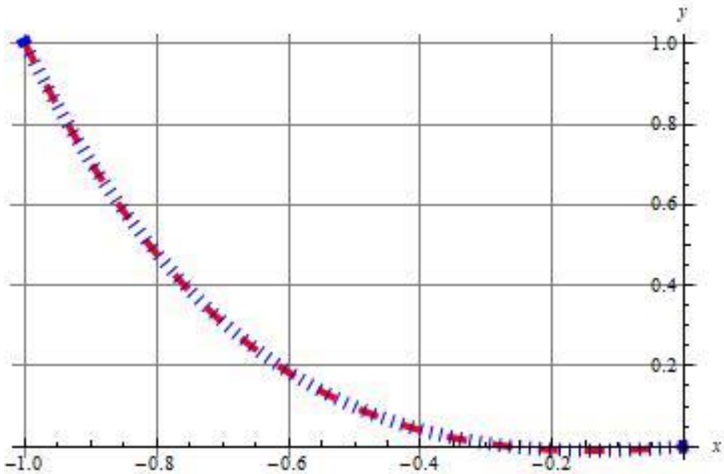

Figure 15. The analytic solution (red dashing) and the DTM-solution (blue dotted).

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CHAPTER 4

AN OVERVIEW ON LINEARIZATION METHOD

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INTRODUCTION

The main purpose of this chapter is to express and use linearization methods to find the solutions of nonlinear equations of different orders and fractional order that cannot be solved using the usual methods or their solutions can be obtained with difficulty. Some examples of the methods that have been researched in this study are: Backlund transformation method of Riccati equation, Kudryashov method and three wave method (See for example (Liu, 2019), (Kumar, 2018), (Ma, 2019), (Zhao, 2020),(Liano, 1992) ,(Wazwaz, 2007),(He, 2006),(Kudryashov, 2016),(Manafian, 2017),(Biswas, 2020),(Sajid, 2020) and (Ghanbari, 2021)). All methods are described in detail and their application is presented with examples. Recently, a new modification of Riemann-Liouville derivative is proposed by Jumarie:

$$D_x^\alpha f(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-\varepsilon)^{-\alpha}\left(f(\varepsilon)-f(0)\right)d\varepsilon, \qquad 0<\alpha<1$$

and gave some basic fractional calculus formulae, for example:

$$D_x^\alpha\left(u(x)v(x)\right)=v(x)D_x^\alpha\left(u(x)\right)+u(x)D_x^\alpha\left(v(x)\right), \tag{a}$$

$$D_x^\alpha\left(f\left(u(x)\right)\right)=f_u'(u)D_x^\alpha\left(u(x)\right)=D_x^\alpha f(u)\left(u_x'\right)^\alpha, \tag{b}$$

The last formula (b) has been applied to solve the exact solutions to some nonlinear fractional order differential equations. If this formula were true, then we could take the transformation $\xi=x-\frac{kt^\alpha}{\Gamma(1+\alpha)}$ and reduce the partial derivative $\frac{\partial^\alpha U(x,t)}{\partial t^\alpha}$ to $U'(\xi)$. Therefore the corresponding fractional differential equations become the ordinary differential equations which are easy to study. But we must point out that Jumarie's basic formulae (a) and (b) are not correct, and therefore the corresponding results on differential equations are not true.

(i) Riemann-Liouville definition: If n is a positive integer and $\alpha\in[n-1,n)$ the α^{th} derivative of f is given by

$$D_a^\alpha f\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\frac{d^n}{dt^n}\int_a^t\frac{f\left(x\right)}{\left(t-x\right)^{\alpha-n+1}}dx$$

(ii) Caputo Definition. For $\alpha\in\left[n-1,n\right)$ the α derivative of f is

$$D_a^\alpha f\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\int_a^t\frac{f^n\left(x\right)}{\left(t-x\right)^{\alpha-n+1}}dx.$$

Now, all definitions are attempted to satisfy the usual properties of the standard derivative. The only property inherited by all definitions of fractional derivative is the linearity property.

Definition 1. Let $f^\alpha\left(t\right)$ stands for $T_\alpha\left(f\right)\left(t\right)$. Hence

$$f^\alpha\left(t\right)=\lim_{\xi\to 0}\frac{f\left(t+\xi t^{1-\alpha}\right)-f\left(t\right)}{\xi}$$

If f is α-differentiable in some $\left(0,a\right)$, $a>0$, and $\lim_{t\to 0^+}f^\alpha\left(t\right)$ exists, then by definition

$$f^\alpha\left(0\right)=\lim_{t\to 0^+}f^\alpha\left(t\right)$$

We should remark that $T_\alpha\left(t^\mu\right)=\mu t^{\mu-\alpha}$. Further, this definition coincides with the classical definitions of R-L and of Caputo on polynomials (up to a constant multiple).

One can easily show that T_α satisfies all the properties in the theorem.

Theorem 2. Let $\alpha\in\left[0,1\right)$ and f,g be α-differentiable at a point t, Then:

$$\left(i\right)T_\alpha\left(af+bg\right)=aT_\alpha\left(f\right)+bT_\alpha\left(g\right), for\quad all\quad a,b\in\mathbb{R}.$$

$$\left(ii\right)T_\alpha\left(t^\mu\right)=\mu t^{\mu-\alpha}, for\quad all\quad \mu\in\mathbb{R}$$

$$\left(iii\right)T_\alpha\left(fg\right)=fT_\alpha\left(g\right)+gT_\alpha\left(f\right)$$

$$\left(iv\right)T_\alpha\left(\frac{f}{g}\right)=\frac{fT_\alpha\left(g\right)-gT_\alpha\left(f\right)}{g^2}$$

If, in addition, f is differentiable, then $T_{\alpha}(f)(t)=t^{1-\alpha}\frac{df}{dt}$.

However, it is worth noting the following fractional derivatives of certain functions:

$$(i)T_{\alpha}\left(e^{\frac{1}{\alpha}t^{\alpha}}\right)=e^{\frac{1}{\alpha}t}.$$

$$(ii)T_{\alpha}\left(\sin\frac{1}{\alpha}t\right)=\cos\frac{1}{\alpha}t,.$$

$$(iii)T_{\alpha}\left(\cos\frac{1}{\alpha}t\right)=-\sin\frac{1}{\alpha}t,.$$

Definition 3. (Fractional Integral) Let $a\geq 0$ and $t\geq a$. Also, let f be a function defined on $(a,t]$ and $\alpha\in f$. Then the $\alpha-$fractional integral of f is defined by,

$$I_{\alpha}^{\alpha}(f)(t)=\int_{a}^{t}\frac{f(x)}{x^{1-\alpha}}dx$$

1- alghoritm of the Backlund transformation method of Riccati equation for NLSE

At first we consider the following Riccati equation:

$$\varphi'(\xi)=\sigma+\varphi^{2}(\xi), \tag{1}$$

which has the following exact solutions

$$\varphi=\begin{cases}-\sqrt{-\sigma}tanh\left(\sqrt{-\sigma}\,\xi\right), & \sigma<0,\\ -\sqrt{-\sigma}\cot h\left(\sqrt{-\sigma}\,\xi\right), & \sigma<0,\\ -\dfrac{1}{\xi+\varpi},\varpi=const. & \sigma=0.\\ \sqrt{\sigma}tan\left(\sqrt{-\sigma}\,\xi\right), & \sigma>0,\\ -\sqrt{\sigma}\cot\left(\sqrt{-\sigma}\,\xi\right), & \sigma>0,\end{cases} \tag{2}$$

Next, let us consider the nonlinear evolution equation (NLEE):

$$G\left(u,D_{t}^{\alpha}u,D_{x}^{\beta}u,D_{y}^{\psi}u,D_{t}^{\alpha}D_{t}^{\alpha}u,D_{t}^{\alpha}D_{x}^{\beta}u,D_{x}^{\beta}D_{x}^{\beta}u,....\right)=0,\qquad 0<\alpha,\beta,\psi<1. \tag{3}$$

Where u is an unknown function, and G is a polynomial of u. In this equation, the partial fractional derivatives involving the highest order derivatives and the nonlinear terms are included. Next by using the new definition for traveling wave variable

$$u(x,t)=U(\xi)e^{i\left(k\frac{x^{\beta}}{\beta}+c\frac{t^{\alpha}}{\alpha}\right)}, \qquad \xi=l\frac{x^{\beta}}{\beta}+\omega\frac{t^{\alpha}}{\alpha} \tag{4}$$

Where k,c,l and ω are non-zero arbitrary constants, we can rewrite Eq. (3) as the following nonlinear ODE:

$$P(U,U',U'',\ ...)=0. \tag{5}$$

Step 1: Suppose that Eq. (5) has the following solution (the main idea of this study)

$$U(\xi)=\sum_{i=0}^{N}A_i\left(m+\psi(\xi)\right)^i+\sum_{i=1}^{N}B_i\left(m+\psi(\xi)\right)^{-i} \tag{6}$$

Where A_i,B_i are constants to be determined and $\psi(\xi)$ comes from the following Backlund transformation for the Riccati equation:

$$\psi(\xi)=\frac{-\sigma B+D\varphi(\xi)}{D+B\ \varphi(\xi)} \tag{7}$$

And $\psi(\xi)$ satisfies in the Riccati equation

$$\psi'(\xi)=\sigma+\psi^2(\xi), \tag{8}$$

Where B,D are arbitrary parameters, σ is a constant to be determined and $B\neq 0,\varphi(\xi)$ are the well-known solutions (2). For simplicity we assume

$$F=m+\psi(\xi) \tag{9}$$

So

$$U(\xi)=\sum_{i=0}^{N}A_iF^i+\sum_{i=1}^{N}B_iF^{-i} \tag{10}$$

and

$$\psi'(\xi) \;=\sigma+\psi^2(\xi)\Rightarrow F'=\psi'(\xi) \;=\sigma+\psi^2(\xi)$$

Step 2 : Balancing the highest order derivatives and nonlinear term in (5) to determine the positive integer N in (6).
Step 3 : Substituting the explicit formal solution (6) with (7) into Eq. (5) and setting the coefficients of the powers of φ(ξ) to be zero, we obtain a system of algebraic equations which can be solved by the Maple or Mathematica to get the unknown constants A_i, B_i, σ, k and c . Consequently, we obtain the exact solutions of Eq. (3). Now we consider the NLSE with group velocity dispersion coefficient and second order spatiotemporal dispersion coefficient as follows

$$i\left(\frac{\partial^\alpha q}{\partial x^\alpha}+\gamma_1\frac{\partial^\alpha q}{\partial t^\alpha}\right)+\gamma_2\frac{\partial^2 q}{\partial t^2}+\gamma_3\frac{\partial^2 q}{\partial x^2}+|q|^2 q=0 \tag{11}$$

In order to extract optical solitons, one may have the following starting hypothesis, the wave profile is split into amplitude and phase components, respectively, as

$$q(x,t)=U(\xi)e^{i\psi} \tag{12}$$

Where $\xi=\gamma_2\left(\frac{x^\alpha}{\alpha}-v\frac{t^\alpha}{\alpha}\right)$. The $U(\xi)$ is the amplitude components of the wave profiles, while the phase factor is given by

$$\psi=-k\frac{x^\alpha}{\alpha}+\omega\frac{t^\alpha}{\alpha}+\upsilon, \tag{13}$$

where _ is the frequency of the solitons while ω represents the wave number and υ is the phase constant. The following section explains the integration scheme. Substituting (2.2) into (1.1), and then decomposing into real and imaginary parts leads to a pair of relations. The imaginary part leads to a constraint relation between the soliton parameters as

$$B=\frac{1-2\gamma_2\kappa}{\gamma_1\upsilon} \tag{14}$$

which is a constraint condition for the solitons to exist. The real part equation is given as

$$\frac{(1-2\beta\kappa)^2\left(\beta v^2+\gamma_3\right)}{\alpha^2\upsilon^2}U''+U^3-\beta\omega^2U^2+\left(\kappa+\alpha\omega+\gamma\kappa^2\right)U=0 \quad (15)$$

By applying the homogeneous balance in (15), we have n = 1. Suppose (15) have the solution of the form

$$U(\xi)=A_0+A_1F+B_1F^{-1} \quad (16)$$

Now substituting Eq. (16) along with Eqs. (12-13) into Eq. (15), we get a polynomial in $F(\xi)$. Equating the coefficient of same power of $F^i(\xi)\left(i=0,\pm1,\pm2,...\right)$, we attain the system of algebraic equations, and by solving these obtained system of equations for A_0, A_1, B_1, m and σ, and by solving obtained system we get the following values:

Family 1:

$$A_0=\frac{1}{3}\beta\omega^2,\ A_1=\sqrt{\frac{-2(1-2\beta\kappa)^2\left(\beta v^2+\gamma_3\right)}{\alpha^2\upsilon^2}},$$

$$B_1=\frac{1}{18}\frac{\alpha^2\upsilon^2\sqrt{\frac{-2(1-2\beta\kappa)^2\left(\beta v^2+\gamma_3\right)}{\alpha^2\upsilon^2}}}{(1-2\beta\kappa)^2\left(\beta v^2+\gamma_3\right)}\times$$
$$\left(\frac{6(1-2\beta\kappa)^2\left(\beta v^2+\gamma_3\right)}{\alpha^2\upsilon^2}\sigma-\beta^2\omega^4+3\left(\kappa+\alpha\omega+\gamma\kappa^2\right)\right).$$

Now by using Eqs. (15)- (17), and substituting the general solutions of Eq. (7) into Eq. (13), we have three types of travelling wave solutions of the time and space fractional derivatives cubic nonlinear Schrodinger as follows:

Solutions for family 1:

When $\sigma < 0$,

$$U(\xi) = \frac{1}{3}\beta\omega^2 + \sqrt{\frac{-2(1-2\beta\kappa)^2(\beta\nu^2+\gamma_3)}{\alpha^2\upsilon^2}} \times$$

$$\left(m + \frac{-\sigma B - D\sqrt{-\sigma}tanh\left(\sqrt{-\sigma}\,\xi\right)}{D - B\sqrt{-\sigma}tanh\left(\sqrt{-\sigma}\,\xi\right)}\right) + \frac{1}{18}\frac{\alpha^2\upsilon^2\sqrt{\frac{-2(1-2\beta\kappa)^2(\beta\nu^2+\gamma_3)}{\alpha^2\upsilon^2}}}{(1-2\beta\kappa)^2(\beta\nu^2+\gamma_3)} \times$$

$$\left(\frac{6(1-2\beta\kappa)^2(\beta\nu^2+\gamma_3)}{\alpha^2\upsilon^2}\sigma - \beta^2\omega^4 + 3\left(\kappa + \alpha\omega + \gamma\kappa^2\right)\right) \times$$

$$\left(m + \frac{-\sigma B - D\sqrt{-\sigma}tanh\left(\sqrt{-\sigma}\,\xi\right)}{D - B\sqrt{-\sigma}tanh\left(\sqrt{-\sigma}\,\xi\right)}\right)^{-1},$$

So

$$q_{1-1}(x,t) = \frac{1}{3}\beta\omega^2 e^{i\left(-k\frac{x^\alpha}{\alpha}+\omega\frac{t^\alpha}{\alpha}+\upsilon\right)} + \sqrt{\frac{-2(1-2\beta\kappa)^2(\beta\nu^2+\gamma_3)}{\alpha^2\upsilon^2}} \times$$

$$\left(m + \frac{-\sigma B - D\sqrt{-\sigma}tanh\left(\sqrt{-\sigma}\,\gamma_2\left(\frac{x^\alpha}{\alpha} - v\frac{t^\alpha}{\alpha}\right)\right)}{D - B\sqrt{-\sigma}tanh\left(\sqrt{-\sigma}\,\xi\right)\gamma_2\left(\frac{x^\alpha}{\alpha} - v\frac{t^\alpha}{\alpha}\right)}\right) \times e^{i\left(-k\frac{x^\alpha}{\alpha}+\omega\frac{t^\alpha}{\alpha}+\upsilon\right)}$$

$$+\frac{1}{18}\frac{\alpha^2\upsilon^2\sqrt{\frac{-2(1-2\beta\kappa)^2(\beta\nu^2+\gamma_3)}{\alpha^2\upsilon^2}}}{(1-2\beta\kappa)^2(\beta\nu^2+\gamma_3)} \times$$

$$\left(\frac{6(1-2\beta\kappa)^2(\beta\nu^2+\gamma_3)}{\alpha^2\upsilon^2}\sigma - \beta^2\omega^4 + 3\left(\kappa + \alpha\omega + \gamma\kappa^2\right)\right) \times$$

$$\left(m + \frac{-\sigma B - D\sqrt{-\sigma}tanh\left(\sqrt{-\sigma}\,\gamma_2\left(\frac{x^\alpha}{\alpha} - v\frac{t^\alpha}{\alpha}\right)\right)}{D - B\sqrt{-\sigma}tanh\left(\sqrt{-\sigma}\,\gamma_2\left(\frac{x^\alpha}{\alpha} - v\frac{t^\alpha}{\alpha}\right)\right)}\right)^{-1} e^{i\left(-k\frac{x^\alpha}{\alpha}+\omega\frac{t^\alpha}{\alpha}+\upsilon\right)},$$

And

$$q_{1-2}(x,t)=\frac{1}{3}\beta\omega^2 e^{i\left(-k\frac{x^\alpha}{\alpha}+\omega\frac{t^\alpha}{\alpha}+\upsilon\right)}+\sqrt{\frac{-2(1-2\beta\kappa)^2(\beta\nu^2+\gamma_3)}{\alpha^2\upsilon^2}}\times$$

$$\left(m+\frac{-\sigma B-D\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\,\gamma_2\left(\frac{x^\alpha}{\alpha}-\nu\frac{t^\alpha}{\alpha}\right)\right)}{D-B\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\,\xi\right)\gamma_2\left(\frac{x^\alpha}{\alpha}-\nu\frac{t^\alpha}{\alpha}\right)}\right)\times e^{i\left(-k\frac{x^\alpha}{\alpha}+\omega\frac{t^\alpha}{\alpha}+\upsilon\right)}$$

$$+\frac{1}{18}\frac{\alpha^2\upsilon^2\sqrt{\frac{-2(1-2\beta\kappa)^2(\beta\nu^2+\gamma_3)}{\alpha^2\upsilon^2}}}{(1-2\beta\kappa)^2(\beta\nu^2+\gamma_3)}\times$$

$$\left(\frac{6(1-2\beta\kappa)^2(\beta\nu^2+\gamma_3)}{\alpha^2\upsilon^2}\sigma-\beta^2\omega^4+3\left(\kappa+\alpha\omega+\gamma\kappa^2\right)\right)\times$$

$$\left(m+\frac{-\sigma B-D\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\,\gamma_2\left(\frac{x^\alpha}{\alpha}-\nu\frac{t^\alpha}{\alpha}\right)\right)}{D-B\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\,\gamma_2\left(\frac{x^\alpha}{\alpha}-\nu\frac{t^\alpha}{\alpha}\right)\right)}\right)^{-1} e^{i\left(-k\frac{x^\alpha}{\alpha}+\omega\frac{t^\alpha}{\alpha}+\upsilon\right)},$$

2- Kudryashov method

In this section we apply the Kudryashov method for solving fractional coupled nonlinear Schrodinger equations

$$\begin{cases} i\dfrac{\partial^\alpha\psi_1}{\partial t^\alpha}+\dfrac{1}{2}\dfrac{\partial^2\psi_1}{\partial x^2}+\left(|\psi_1|^2+e|\psi_2|^2\right)\psi_1=0, \\ i\dfrac{\partial^\alpha\psi_2}{\partial t^\alpha}+\dfrac{1}{2}\dfrac{\partial^2\psi_2}{\partial x^2}+\left(e|\psi_1|^2+|\psi_2|^2\right)\psi_2=0, \end{cases} \qquad -\infty<x<\infty, 0<\alpha\le 1, \quad (17)$$

where ψ_1 and ψ_2 are the wave amplitudes in two polarizations and e is the cross-phase modulation coefficient. The nonlinear Schrodinger equation is an example of a universal nonlinear model that describes many physical nonlinear systems. The equation can be applied to hydrodynamics, nonlinear optics, nonlinear acoustics, quantum condensates, heat pulses in solids and various other nonlinear instability phenomena. Such equations have been shown to govern pulse propagation along orthogonal polarization axes in nonlinear optical fibers and in wavelength-division-multiplexed systems. These equations also model beam propagation inside crystals or photorefractive as well as water wave interactions. Solitary waves in these equations are often called vector solitons in the literature as they generally contain two components. In all the above physical situations, collision of vector solitons is an important issue.

2-1 Analysis of the Method

The purpose of this section is to present the algorithm of the Kudryashov method to find exact solutions of the nonlinear evolution equations. Let us consider the nonlinear partial differential equation in the form

$$G\left(u, D_t^{\alpha}u, D_x^{\beta}u, D_t^{\alpha}D_t^{\alpha}u, D_t^{\alpha}D_x^{\beta}u, D_x^{\beta}D_x^{\beta}u, ...\right)=0, \qquad 0<\alpha,\beta<1. \tag{18}$$

Where u is an unknown function, and G is a polynomial of u. In this equation, the partial fractional derivatives involving the highest order derivatives and the nonlinear terms are included. Next by using the new definition for traveling wave variable

$$u(x,t)=U(\xi)e^{i\left(k\frac{x^{\beta}}{\beta}+c\frac{t^{\alpha}}{\alpha}\right)}, \qquad \xi = l\frac{x^{\beta}}{\beta}+\omega\frac{t^{\alpha}}{\alpha}, \tag{19}$$

where k, c, l and ω are non-zero arbitrary constants, we can rewrite Eq. (18) as the following nonlinear ODE:

$$Q\left(U, U', U'', U''', ...\right)=0, \tag{20}$$

where the prime denotes the derivation with respect to ξ. If possible, we should integrate Eq. (20) term by term one or more times. Now we show how one could obtain the exact solution of the Eq. (20) using the approach by modified Kudryashov method.

2-2 Determination of the dominant term

To find dominant terms we substitute

$$U=\xi^{p}, \tag{21}$$

into all terms of Eq. (20). Then we compare degrees of all terms in Eq. (20) and choose two or more with the smallest degree. The minimum value of P define the pole of solution for Eq. (20) and we denote it as N. We have to point out that method can be applied when N is integer. If the value N is noninteger one can transform the equation not only study the procedure but also repeat it.

2-3 The solution structure

We look for exact solution of Eq. (20) in the form

$$U = a_0 + a_1 Q(\xi) + a_2 Q^2(\xi) + ... + a_N Q^N(\xi), \tag{22}$$

where a_i are unknown constants to be determined later, such that $a_N \neq 0$, while $Q(\xi)$ have the form

$$Q(\xi) = \frac{1}{1+e^{\xi}}. \tag{23}$$

These functions satisfies to the first order ordinary differential equations (Riccati equations)

$$Q'(\xi) = Q^2(\xi) - \mathrm{Q}(\xi). \tag{24}$$

Eqs. (22) are necessary to calculate the derivatives of functions $Q(\xi)$.

Remark : This Riccati equation also admits the following exact solutions:

$$\begin{aligned} Q_1(\xi) &= \frac{1}{2}\left(1 - \tanh\left[\frac{\xi}{2} - \frac{\varepsilon \ln \xi_0}{2}\right]\right), \quad \xi_0 > 0, \\ Q_2(\xi) &= \frac{1}{2}\left(1 - \coth\left[\frac{\xi}{2} - \frac{\varepsilon \ln \xi_0}{2}\right]\right), \quad \xi_0 < 0. \end{aligned} \tag{25}$$

2-4 Application to the fractional coupled nonlinear Schrodinger equations

For our purpose, we introduce the following transformations:

$$\psi_1 = y_1(\xi)e^{i\eta} \qquad \xi = x + \omega\frac{t^{\alpha}}{\alpha}, \eta = x + \gamma\frac{t^{\alpha}}{\alpha}, \tag{26}$$
$$\psi_2 = y_2(\xi)e^{i\eta}$$

where ω and γ are constants. By substituting Eqs. (26) into Eqs. (1), is reduced into an ODE

$$\frac{1}{2}y_1'' + i(\omega+1)y_1' - \left(\gamma + \frac{1}{2}\right)y_1 + y_1^3 + ey_1y_2^2 = 0, \tag{27}$$

$$\frac{1}{2}y_2'' + i(\omega+1)y_2' - \left(\gamma + \frac{1}{2}\right)y_2 + y_2^3 + ey_1^2y_2 = 0, \tag{28}$$

where $' = \frac{d}{d\xi}$. By using the ansatz [20], the pole order of Eqs. (27) and (28) are $m_1 = 1, m_2 = 1$. So we look for solution of Eqs. (14) and (15) in the following form

$$y_1 = a_0 + a_1\phi, \tag{29}$$

$$y_2 = b_0 + b_1\phi. \tag{30}$$

Substituting Eqs. (29) and (30) into Eqs. (27) and (28), we obtain the system of algebraic equations and solving these algebraic equations we have:

Case1:

$$a_1 = \sqrt{\begin{matrix}\left(\dfrac{-8e^2a_0^3 - 3i\left(-\frac{1}{2}ie\left(3ea_0 \pm \sqrt{9e^2a_0^2 - 32a_0^2 + 6a_0 - 12a_0^3}\right)+1\right)}{4e^2a_0^3}\right) \\ -\left(\left(-\frac{1}{2}ie\left(3ea_0 \pm \sqrt{9e^2a_0^2 - 32a_0^2 + 6a_0 - 12a_0^3}\right)+1\right)^2 - 9/4\right)\end{matrix}},$$

$$b_1=\sqrt{\frac{\left(\dfrac{4e^2a_0^3-3i\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)+1\right)}{4e^3a_0^3}\right)}{-\left(\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)+1\right)^2-9/4\right)}},\tag{31}$$

$$b_0=0,$$

$$\omega=-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right),$$

$$\gamma=-\frac{1}{2}+a_0^2.$$

In this case a_0 is arbitrary. By using this advantages (31) into Eqs. (29) and (30) along with (25) we have

$$y_1=a_0+\sqrt{\frac{\dfrac{-8e^2a_0^3-3i\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)+1\right)}{4e^2a_0^3}}{-\left(\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)+1\right)^2-9/4\right)}}$$

$$\times\left[1+e^{x+\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)\right)\frac{t^\alpha}{\alpha}}\right]^{-1},$$

$$y_2=\sqrt{\frac{\dfrac{4e^2a_0^3-3i\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)+1\right)}{4e^3a_0^3}}{-\left(\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)+1\right)^2-9/4\right)}}$$

$$\times\left[1+e^{x+\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)\right)\frac{t^\alpha}{\alpha}}\right]^{-1}.$$

So from (26) we obtain solitary wave solutions of Eqs. (1) as follow

$$\psi_{1,1}(x,t)=\left[a_0+\sqrt{\left(\frac{-8e^2a_0^3-3i\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)+1\right)}{4e^2a_0^3}\right)-\left(\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)+1\right)^2-9/4\right)}\right.$$

$$\left.\times\left[1+e^{x+\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)\right)\frac{t^\alpha}{\alpha}}\right]^{-1}\right]e^{i\left(x+\left(-\frac{1}{2}+a_0^2\right)\frac{t^\alpha}{\alpha}\right)},$$

$$\psi_{2,1}(x,t)=\sqrt{\left(\frac{4e^2a_0^3-3i\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)+1\right)}{4e^3a_0^3}\right)-\left(\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)+1\right)^2-9/4\right)}$$

$$\times\left[1+e^{x+\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)\right)\frac{t^\alpha}{\alpha}}\right]^{-1}e^{i\left(x+\left(-\frac{1}{2}+a_0^2\right)\frac{t^\alpha}{\alpha}\right)}.$$

Cas2:

$$a_1=\sqrt{\frac{-4ea_0-4a_0+3i\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)-\frac{1}{2}i-a_0^2\right)}{4e^2a_0}},$$

$$b_1=\sqrt{\frac{4a_0-3i\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)-\frac{1}{2}i-a_0^2\right)}{4ea_0}}, \tag{32}$$

$$b_0=0,$$

$$\omega=-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right),$$

$$\gamma=-\frac{1}{2}+a_0^2.$$

In this case a_0 is arbitrary and by substituting relations (32) into Eqs. (29) and (30) along with (25) we have solutions of Eqs. (1) as follow

$$\psi_{1,2}(x,t)=\left[a_0+\sqrt{\frac{-4ea_0-4a_0+3i\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)-\frac{1}{2}i-a_0^2\right)}{4e^2a_0}}\right.$$
$$\left.\times\left[1+e^{x+\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)\right)\frac{t^\alpha}{\alpha}}\right]^{-1}\right]e^{i\left(x+\left(-\frac{1}{2}+a_0^2\right)\frac{t^\alpha}{\alpha}\right)},$$

$$\psi_{2,2}(x,t)=\sqrt{\frac{4a_0-3i\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)-\frac{1}{2}i-a_0^2\right)}{4ea_0}}$$
$$\times\left[1+e^{x+\left(-\frac{1}{2}ie\left(3ea_0\pm\sqrt{9e^2a_0^2-32a_0^2+6a_0-12a_0^3}\right)\right)\frac{t^\alpha}{\alpha}}\right]^{-1}e^{i\left(x+\left(-\frac{1}{2}+a_0^2\right)\frac{t^\alpha}{\alpha}\right)}.$$

3- Time fractional coupled Boussinesq equation

In following we consider the following time fractional coupled Boussinesq equation (BE)

$$\begin{cases}\frac{\partial^\alpha u}{\partial t^\alpha}+uu_x+v_x+qu_{xx}=0\\ \frac{\partial^\alpha v}{\partial t^\alpha}+(uv)_x+pu_{xxx}-qv_{xx}=0\end{cases},0\le\alpha<1. \tag{33}$$

Where $p,q\in R$. Using the transformation $u(x,t)=U(\xi),v(x,t)=V(\xi)$; where $\xi=kx+\omega\frac{t^\alpha}{\alpha}$ and once integrating respect to ξ, Eq. (33) becomes an following ordinary differential equation,

$$\begin{aligned} &\omega U + \frac{k}{2}U^2 + kV + qk^2U' = R_1, \\ &\omega V + kUV + pk^3U'' - qk^2V' = R_2, \end{aligned} \tag{34}$$

Where R_1 and R_2 are the integration constants of first- and second-equation of system (34), respectively. From first-equation of system (2), we get

$$V = \frac{1}{k}\left(R_1 - \omega U - \frac{k}{2}U^2 - qkU' \right), \tag{35}$$

By substituting Eq. (35) into the second-equation of system (34), and for simplifying we set $R_1 = 0$ and $R_2 = 0$, we get the following covering equation

$$-\frac{\omega^2}{k}U - \frac{3}{2}\omega U^2 - \frac{k}{2}U^3 + k^3\left(p + q^2\right)U'' = 0, \tag{36}$$

A brief application of the method to the FBE is provided in the second section of this paper. In section three graphical behavior of solutions introduced. Finally, conclusions are presented in the last section of the article.

4. *Three wave approaches to the FBE*

Primis, we suppose that Eq. (9) has the following three-wave solutions

$$U(\xi) = \gamma_1 e^{\delta\xi} + \gamma_2 \cos(\lambda_1\xi) + \gamma_3 e^{-\delta\xi} + 2\gamma_4 \cosh(\lambda_2\xi). \tag{37}$$

Where $g_1, .., g_4, d, l_1, l_2$ are unfamiliar constants to be determined later. With substituting (41) into (36) and collect coefficients of $e^{i\delta\xi}, \cos(\lambda_1\xi), \cosh(\lambda_2\xi), \sin(\lambda_1\xi), \sinh(\lambda_2\xi)$, $i = -2, -1, 0, 1, 2$ and let them equal to zero. So we obtain the algebraic equations and by solving these equations we have:

Set 1: $\gamma_1 = 0, \gamma_3 = 0, \gamma_2 \neq 0, \gamma_4 \neq 0$ then by solving algebraic equation we have

$$\gamma_2=\frac{1}{2},\gamma_4=-\frac{1}{2},w=-\frac{1}{2}\sqrt{q^2+pk^2},k=k \tag{38}$$

So we have general solutions of eq. (33) as follows

$$u_1(x,t)=\frac{1}{2}\cos\left(\lambda_1\left(kx-\frac{1}{2}\sqrt{q^2+pk^2}\,\frac{t^\alpha}{\alpha}\right)\right)-\cosh\left(\lambda_2\left(kx-\frac{1}{2}\sqrt{q^2+pk^2}\,\frac{t^\alpha}{\alpha}\right)\right)$$

So from (35) we directly obtain

$$v_1(x,t)=\left[R_1-\frac{w}{2}\cos\left(\lambda_1\left(kx-\frac{1}{2}\sqrt{q^2+pk^2}\,\frac{t^\alpha}{\alpha}\right)\right)+w\cosh\left(\lambda_2\left(kx-\frac{1}{2}\sqrt{q^2+pk^2}\,\frac{t^\alpha}{\alpha}\right)\right)-\right.$$
$$\frac{k}{2}\left(\frac{1}{2}\cos\left(\lambda_1\left(kx-\frac{1}{2}\sqrt{q^2+pk^2}\,\frac{t^\alpha}{\alpha}\right)\right)-\cosh\left(\lambda_2\left(kx-\frac{1}{2}\sqrt{q^2+pk^2}\,\frac{t^\alpha}{\alpha}\right)\right)\right)^2+$$
$$\left.\frac{1}{2}qk\sin\left(\lambda_1\left(kx-\frac{1}{2}\sqrt{q^2+pk^2}\,\frac{t^\alpha}{\alpha}\right)\right)+qk\sinh\left(\lambda_2\left(kx-\frac{1}{2}\sqrt{q^2+pk^2}\,\frac{t^\alpha}{\alpha}\right)\right)\right]$$

Set 2: $\gamma_2=0,\gamma_4=0,\gamma_1\neq 0,\gamma_3\neq 0$ then by solving algebraic equation we have

$$\gamma_1=\frac{2}{3}\frac{\delta^2k^4q^2+\delta^2k^4p-w^2}{k^2\gamma_3},\gamma_3=\gamma_3,w=\sqrt{q^2+pk^2}\delta,k=k \tag{39}$$

So we have

$$u_2(x,t)=\frac{2}{3}\frac{\delta^2k^4q^2+\delta^2k^4p-w^2}{k^2\gamma_3}e^{\delta\left(kx+\sqrt{q^2+pk^2}\delta\frac{t^\alpha}{\alpha}\right)}+\gamma_3e^{-\delta\left(kx+\sqrt{q^2+pk^2}\delta\frac{t^\alpha}{\alpha}\right)}]$$

From (39) we have

$$v_2(x,t) = \frac{2}{3}\frac{\delta^2 k^4 q^2 + \delta^2 k^4 p - w^2}{k^3 \gamma_3} e^{\delta\left(kx + \sqrt{q^2+pk^2}\delta\frac{t^\alpha}{\alpha}\right)} + \frac{\gamma_3}{k} e^{-\delta\left(kx + \sqrt{q^2+pk^2}\delta\frac{t^\alpha}{\alpha}\right)} -$$

$$-\frac{1}{2}\left(\frac{2}{3}\frac{\delta^2 k^4 q^2 + \delta^2 k^4 p - w^2}{k^2 \gamma_3} e^{\delta\left(kx + \sqrt{q^2+pk^2}\delta\frac{t^\alpha}{\alpha}\right)} + \gamma_3 e^{-\delta\left(kx + \sqrt{q^2+pk^2}\delta\frac{t^\alpha}{\alpha}\right)}\right)^2 -$$

$$q\left(\frac{2}{3}\delta\frac{\delta^2 k^4 q^2 + \delta^2 k^4 p - w^2}{k^2 \gamma_3} e^{\delta\left(kx + \sqrt{q^2+pk^2}\delta\frac{t^\alpha}{\alpha}\right)} - \gamma_3 \delta e^{-\delta\left(kx + \sqrt{q^2+pk^2}\delta\frac{t^\alpha}{\alpha}\right)}\right)$$

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